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AND APPLIED SCIENCE



COMPUTATIONAL EXPERIENCE WITH OPTIMAL  
VALUE FUNCTION AND LAGRANGE  
MULTIPLIER SENSITIVITY  
IN NLP

by

Robert L. Armacost

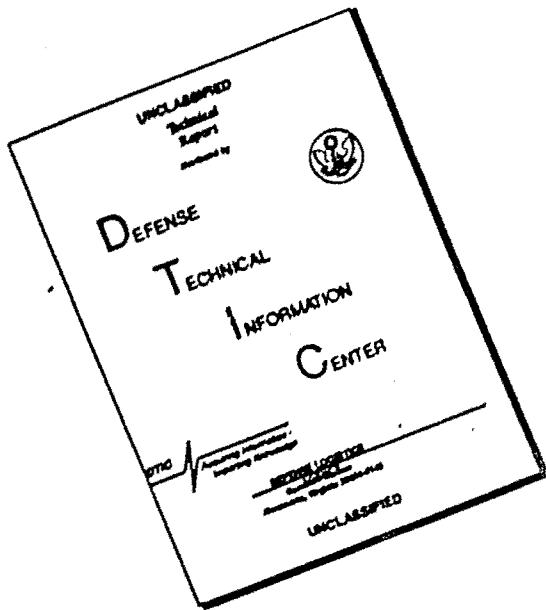
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20. Abstract

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\*Presently assigned to Coast Guard Headquarters, Washington, D.C. The opinions or assertions contained herein are the private ones of the author and are not to be construed as official or reflecting the views of the Commandant or the Coast Guard at large.

## TABLE OF CONTENTS

1. Introduction . . . . .	1
2. Supporting Theory . . . . .	3
3. User Options and Computer Codes . . . . .	8
4. New Computational Experience . . . . .	10
5. Large-Scale, Multi-Item Inventory Model . . . . .	30
6. Conclusions . . . . .	40
References . . . . .	42
Appendix A . . . . .	43
Appendix B . . . . .	48
Appendix C . . . . .	50

LIST OF TABLES

Table	Page
1. The Second Option Card . . . . .	9
2. Trajectory Results for Problem B . . . . .	12
3. Trajectory Results for Problem C . . . . .	16
4. First Order Sensitivity Comparison . . . . .	24
5. Variation in the Components of $H_p$ . . . . .	27
6. Multi-Item Inventory Problem Data . . . . .	33

## LIST OF ILLUSTRATIONS

Figure	Page
1. Partial Derivatives for Problem B . . . . .	13
2. Shell Primal Subproblem and Sensitivity Output . . . . .	19
3. Shell Dual Subproblem and Sensitivity Output . . . . .	22
4. Sensitivity Results for Problem F . . . . .	29
5. Computer Output for Schrady-Choe Inventory Problem . . . . .	34
6. Computer Output for Perturbed Schrady-Choe Problem . . . . .	37

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COMPUTATIONAL EXPERIENCE WITH OPTIMAL VALUE FUNCTION  
AND LAGRANGE MULTIPLIER SENSITIVITY IN NLP

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1. Introduction

Several aspects of sensitivity analysis in nonlinear programming have been examined from a computational viewpoint by Armacost and Fiacco (1974). That work was based on the theory developed by Fiacco (1973) and used the computational procedures implemented by Armacost and Mylander (1973) using the SUMT-Version 4 computer code with the logarithmic-quadratic loss penalty function modified to estimate the partial derivatives of the solution point and the objective function taken with respect to certain specified problem parameters. Fiacco (1973) developed the necessary formulas to provide estimates of the partial derivatives of the Lagrange multipliers taken with respect to the problem parameters using the logarithmic-quadratic loss penalty function. Armacost and Fiacco (1975) developed the formulas to obtain first and second order sensitivity information for the optimal value function (a function of the parameters defined by the objective function evaluated at the solution point). Additionally, Armacost and Fiacco (1976) have shown that when the parameters are the right-hand side components of the constraints, the partial derivatives of the Lagrange multipliers are the components of the Hessian of the optimal value function. The supporting theory is addressed in Section 2.

Armacost and Fiacco (1974) focus on the computational experience with four example problems discussing various aspects of the sensitivity analysis procedure and results. They note that for large problems with a large number of parameters, a very large number of partial derivatives will be estimated. This is not only time consuming, but is also burdensome to the user who must evaluate all of them, many of which are zero or very close to zero in value. In addition, it is unlikely that the value of the solution (the value of the objective function evaluated at the solution point) will be sensitive to more than a relatively few parameters. Because of this, the method developed by Armacost and Fiacco (1975) to estimate the first order sensitivity of the optimal value function is incorporated in the computer program here to provide an option for preliminary screening of the parameters to determine which ones affect the optimal value function. Using the formulas developed by Fiacco (1973), a second option is included which permits the calculation of the sensitivity estimates for the Lagrange multipliers. The computer code and options used to accomplish these calculations are discussed in Section 3.

In Section 4, the new computational experience with Lagrange multiplier and optimal value function sensitivity using these options is presented for the four examples of Armacost and Fiacco (1974).

In Section 5, a sensitivity analysis is conducted for a large scale, multi-item inventory model developed by Schrady and Choe (1971) for the U. S. Navy. While the example used is the same small one used by Schrady and Choe, it nonetheless exhibits the value of performing such a sensitivity analysis in real world situations and illustrates the care that must be taken in interpreting the sensitivity results.

## 2. Supporting Theory

The problems considered here are of the form of Problem  $P(\epsilon)$ .

$$\begin{aligned} & \text{minimize} && f(x, \epsilon) \\ & \text{subject to} && g_i(x, \epsilon) \geq 0, \quad i=1, \dots, m, \\ & && h_j(x, \epsilon) = 0, \quad j=1, \dots, p. \end{aligned}$$

When certain assumptions are satisfied, Fiacco (1973) and Armacost and Fiacco (1975) have shown the existence of the first order sensitivity of a Kuhn-Tucker triple and the first and second order sensitivity of the optimal value function. Additionally, they provide the means of estimating this sensitivity by way of the logarithmic-quadratic loss penalty function. The following four assumptions are sufficient to establish these results and are assumed to hold throughout this Section.

A1 --The functions defining Problem  $P(\epsilon)$  are twice continuously differentiable in  $(x, \epsilon)$  in a neighborhood of  $(x^*, 0)$ .

A2 --The second order sufficient conditions for a local minimum of Problem  $P(0)$  hold at  $x^*$  with associated Lagrange multipliers  $u^*$  and  $w^*$ .

A3 --The gradients  $\nabla_x g_i(x^*, 0)$  for all  $i$  such that  $g_i(x^*, 0) = 0$ , and  $\nabla_x h_j(x^*, 0)$ ,  $j=1, \dots, p$  are linearly independent.

A4 --Strict complementary slackness holds at  $(x^*, 0)$  (i.e.,  $u_i^* > 0$  for all  $i$  such that  $g_i(x^*, 0) = 0$ ).

The main results are presented without proof and are stated here for completeness. The portion of the theory used in the computational algorithm is made specific. The Lagrangian for Problem  $P(\epsilon)$  is

$$L(x, u, w, \epsilon) = f(x, \epsilon) - \sum_{i=1}^m u_i g_i(x, \epsilon) + \sum_{j=1}^p w_j h_j(x, \epsilon)$$

where  $u_i$ ,  $i=1, \dots, m$  and  $w_j$ ,  $j=1, \dots, p$  are the Lagrange multipliers

associated with the inequality and equality constraints respectively.

The first result was proved as Theorem 2.1 by Fiacco (1973).

**THEOREM 1:** (First order sensitivity of a Kuhn-Tucker triple)

If assumptions A1, A2, A3 and A4 hold for Problem P( $\epsilon$ ) at  $(x^*, 0)$ , then

- (a)  $x^*$  is a local isolated minimizing point of Problem P(0) and the associated Lagrange multipliers  $u^*$  and  $w^*$  are unique;
- (b) for  $\epsilon$  in a neighborhood of 0, there exists a unique, once continuously differentiable vector function  $y(\epsilon) = (x(\epsilon), u(\epsilon), w(\epsilon))^T$  satisfying the second order sufficient conditions for a local minimum of problem  $r(\cdot)$  such that  $(x(0), u(0), w(0)) = (x^*, u^*, w^*)$  and hence,  $x(\epsilon)$  is a locally unique, local minimum of Problem P( $\epsilon$ ) with associated unique Lagrange multipliers  $u(\epsilon)$  and  $w(\epsilon)$ ; and
- (c) for  $\epsilon$  near 0, the set of binding inequalities is unchanged, strict complementary slackness holds for  $u_i(\epsilon)$  for  $i$  such that  $g_i(x(\epsilon), \epsilon) = 0$ , and the binding constraint gradients are linearly independent at  $x(\epsilon)$ .

Let  $y(\epsilon) = (x(\epsilon), u(\epsilon), w(\epsilon))^T$  be a Kuhn-Tucker triple where  $x(\epsilon)$  solves problem P( $\epsilon$ ), then the optimal value function is defined as  $f^*(\epsilon) = r(y(\epsilon), \epsilon)$  and the optimal value Lagrangian is  $L^*(\epsilon) = L(x(\epsilon), u(\epsilon), w(\epsilon), \epsilon)$ .

The second result was recently established by Armacost and Fiacco (1973) in their Theorem 2, stated here as Theorem 2.

**THEOREM 2:** (First and second order changes in the optimal value function)

If assumptions A1, A2, A3 and A4 hold for Problem P( $\epsilon$ ) at  $(x^*, 0)$ , then

for  $\epsilon$  near 0,

$$(a) f^*(\epsilon) = L^*(\epsilon);$$

$$(b) \quad \partial_\epsilon f^*(\epsilon) = \partial_\epsilon L(x^*, u^*, w^*, \epsilon) \Big|_{(x^*, u^*, w^*) \sim (x(\epsilon), u(\epsilon), w(\epsilon))}$$

$$\begin{aligned}
 &= \nabla_{\epsilon} f(x, \epsilon) - \sum_{i=1}^m u_i \nabla_{\epsilon} g_i(x, \epsilon) \\
 &\quad + \sum_{j=1}^p w_j \nabla_{\epsilon} h_j(x, \epsilon) \Big|_{(x, u, w) = (x(\epsilon), u(\epsilon), w(\epsilon))} ;
 \end{aligned}$$

$$(c) \quad \nabla_{\epsilon}^2 f^*(\epsilon) = \nabla_{\epsilon} (\nabla_{\epsilon} L(x(\epsilon), u(\epsilon), w(\epsilon), \epsilon)^T) .$$

The problems in subsequent Sections are solved using the logarithmic-quadratic loss penalty function  $W(x, \epsilon, r)$  defined as

$$W(x, \epsilon, r) = f(x, \epsilon) - r \sum_{i=1}^m \ln g_i(x, \epsilon) + (1/2r) \sum_{j=1}^p h_j(x, \epsilon)^2. \quad (1)$$

The following result was obtained by Fiacco (1973) as Theorem 3.1.

**THEOREM 3:** (Approximation of first order sensitivity results and determination of estimates from  $W(x, \epsilon, r)$ )

If assumptions A1, A2, A3 and A4 hold for Problem P( $\epsilon$ ), then for  $(\epsilon, r)$  near  $(0, 0)$ , there exists a locally unique, once continuously differentiable vector function  $y(\epsilon, r) = (x(\epsilon, r), u(\epsilon, r), w(\epsilon, r))^T$  satisfying

$$\begin{aligned}
 \nabla_x L(x, u, w, \epsilon) &= 0, \\
 u_i g_i(x, \epsilon) &= r, \quad i=1, \dots, m, \\
 h_j(x, \epsilon) &= w_j r, \quad j=1, \dots, p,
 \end{aligned}$$

with  $(x(0, 0), u(0, 0), w(0, 0)) = (x^*, u^*, w^*)$ , and such that for any  $(\epsilon, r)$  near  $(0, 0)$  and  $r > 0$ ,  $x(\epsilon, r)$  is a locally unique unconstrained minimizing point of  $W(x, \epsilon, r)$ , with  $u_i(x(\epsilon, r), \epsilon) > 0$ ,  $i=1, \dots, m$ , and  $\nabla_x^2 W(x(\epsilon, r), \epsilon, r)$  is positive definite.

Since the system of equations in Theorem 3 is identically equal to zero at  $r = 0$ , it follows that  $\nabla_{\epsilon} y(\epsilon, r)$  can be calculated for  $(\epsilon, r)$  near  $(0, 0)$ . The following result was shown by Fiacco (1973) following his Theorem 3.1.

**COROLLARY 3.1:** (Convergence of estimates using  $W(x, \epsilon, r)$ )

If assumptions A1, A2, A3 and A4 hold for Problem P( $\epsilon$ ), then for any  $\epsilon$

near 0,

- (a)  $\lim_{\substack{r \rightarrow 0^+ \\ r > 0}} y(\epsilon, r) = y(\epsilon, 0) = y(\epsilon)$ , the Kuhn-Tucker triple characterized in Theorem 1; and
- (b)  $\lim_{\substack{r \rightarrow 0^+ \\ r > 0}} \nabla_\epsilon y(\epsilon, r) = \nabla_\epsilon y(\epsilon, 0) = \nabla_\epsilon y(\epsilon)$ .

Armacost and Fiacco (1974) reported computational experience with sensitivity analysis in four sample nonlinear programming problems. The algorithm uses the fact that the Hessian of the penalty function is positive definite for  $r$  small enough and that the gradient of the penalty function is identically zero in a neighborhood of the solution point. Thus, the gradient of the solution point taken with respect to the parameter vector  $\epsilon$  is estimated as

$$\nabla_\epsilon x(\epsilon, r) = -\nabla_x^2 w(x, \epsilon, r)^{-1} \nabla_\epsilon \nabla_x^2 w(x, \epsilon, r) \Big|_{x=x(\epsilon, r)}. \quad (2)$$

Using the fact that  $u_i(\epsilon, r) = r/g_i(x(\epsilon, r), \epsilon)$  and  $w_j(\epsilon, r) = (1/r)h_j(x(\epsilon, r), \epsilon)$ , the chain rule can be applied to obtain  $\nabla_\epsilon u_i(\epsilon, r)$  and  $\nabla_\epsilon w_j(\epsilon, r)$  as shown by Fiacco (1973). Convergence was shown by Fiacco (1973) following his Corollary 3.1. The above approach is equivalent to calculating  $\nabla_\epsilon y(\epsilon, r)$  directly from the system of equations in Theorem 3.

The logarithmic-quadratic loss penalty function can also be used to provide estimates of the first and second order sensitivity of the optimal value function. Let the optimal value penalty function be defined as  $W^*(\epsilon, r) = W(x(\epsilon, r), \epsilon, r)$ . The first order portion of the sensitivity results developed by Armacost and Fiacco (1975) in their Theorem 4 and Corollary 4.1 follow.

**THEOREM 4:** (First order sensitivity of  $W^*(\epsilon, r)$  and estimates for  $f^*(\epsilon)$ )  
 If assumptions A1, A2, A3 and A4 hold for problem  $P(\epsilon)$ , then for  $(\epsilon, r)$  near  $(0, 0)$  and  $r > 0$ ,

$$(a) \lim_{r \rightarrow 0^+} W^*(\epsilon, r) = L^*(\epsilon) = f^*(\epsilon);$$

$$(b) \nabla_\epsilon W^*(\epsilon, r) = \nabla_\epsilon L(x, u, w, \epsilon) \Big|_{(x, u, w) = (x(\epsilon, r), u(\epsilon, r), w(\epsilon, r))}; \text{ and } (3)$$

$$(c) \lim_{r \rightarrow 0^+} \nabla_\epsilon W^*(\epsilon, r) = \nabla_\epsilon L(x(\epsilon), u(\epsilon), w(\epsilon), \epsilon) = f^*(\epsilon).$$

Another estimate of the optimal value function is obtained as  $f^\#(\epsilon, r) \approx f(x(\epsilon, r), \epsilon)$ . Direct application of the chain rule for differentiation then yields an estimate of the first order sensitivity of the optimal value function as

$$\nabla_\epsilon f^\#(\epsilon, r) = \nabla_x f(x, \epsilon) \nabla_\epsilon x(\epsilon, r) + \nabla_\epsilon f(x, \epsilon). \quad (4)$$

Under the given assumptions, continuity assures that  $f^\#(\epsilon, r) \rightarrow f^*(\epsilon)$  and  $\nabla_\epsilon f^\#(\epsilon, r) \rightarrow \nabla_\epsilon f^*(\epsilon)$  as  $r \rightarrow 0^+$ . Thus, both  $\nabla_\epsilon f^\#(\epsilon, r)$  and  $\nabla_\epsilon W^*(\epsilon, r)$  are estimates of  $\nabla_\epsilon f^*(\epsilon)$  for  $r$  sufficiently small. It is beyond the scope of this Section to explore the relationship between these estimates.

It is easily shown, however, that

$$\begin{aligned} \nabla_\epsilon W^*(\epsilon, r) &= \nabla_\epsilon f^\#(\epsilon, r) - \sum_{i=1}^m u_i (\nabla_x g_i \nabla_\epsilon \lambda(\epsilon, r) + \nabla_\epsilon g_i) \\ &\quad + \sum_{j=1}^p w_j (\nabla_x h_j \nabla_\epsilon x(\epsilon, r) + \nabla_\epsilon h_j) \Big|_{x=x(\epsilon, r)}. \end{aligned}$$

It is easily shown that the terms in the summations on the right approach zero as  $r$  approaches zero. Armacost and Fiacco (1974) used equations (2) and (4) to examine the trajectory and convergence properties of the gradients of the solution point and the optimal value function from a computational point of view. Here, equation (3) is also used to estimate the first order sensitivity of the optimal value function and has the advantage that  $\nabla_x(\epsilon, r)$  need not be calculated.

### 3. User Options and Computer Codes

The basic SUMT-Version 4 computer program and instructions for its use are described in Mylander, Holmes and McCormick (1971). The basic sensitivity analysis subroutines, user instructions, and instructions for integrating the sensitivity package with the SUMT-Version 4 code are described in Armacost and Mylander (1973). Briefly, the conduct of a sensitivity analysis is controlled by the variable NEXOP3 which is given a value on the "Second Option Card" in the SUMT input data deck. There are four choices: no sensitivity analysis, a sensitivity analysis at the final subproblem, a sensitivity analysis at each subproblem along the penalty function minimizing trajectory, or a sensitivity analysis at the final subproblem for a range of differencing increments. In conjunction with this option, two additional options are added here and come into play whenever a sensitivity analysis is conducted. The first option (technically Option 4) is controlled by the variable NEXOP4 and determines whether the partial derivatives of the Lagrange multipliers will be calculated. When the calculation is done, the formulas described by Fiacco (1973) are used. The second option added here (Option 5) permits a screening of the parameters to reduce the number of partial derivatives which are estimated by limiting further analysis to those parameters which will affect the optimal value of the objective function by an amount exceeding 0.1 percent of its current value. This option is controlled by the variable NEXOP5. The estimate of sensitivity of the optimal value function with respect to a particular parameter under this option is calculated using the Armacost and Fiacco (1975) result which involves the partial derivative of the Lagrangian taken with respect to the parameter under consideration. Subroutines LMULT and

PRESEN and related coding in Subroutine SENS implement Option 4 and Option 5, respectively. Subroutines SENS, LMULT and PRESEN are listed in Appendix A. Specific instructions for using these two options in conjunction with the "Second Option Card" are given below in Table 1. This information should be added to Table 5 in Mylander, Holmes and McCormick (1971).

TABLE 1  
THE SECOND OPTION CARD

Option	Column	Value	Meaning
4	28	=0	Do not estimate the partial derivatives of the estimates of the Lagrange multipliers.
		-1	Estimate the partial derivatives of the estimates of the Lagrange multipliers whenever a sensitivity analysis of the solution point is conducted.
5	35	=0	Estimate the partial derivatives of the optimal value function and eliminate those parameters which do not affect the optimal value function from subsequent sensitivity calculations.
		-1	Estimate the partial derivatives of the optimal value function with respect to all parameters, but continue all subsequent sensitivity calculations with respect to all parameters.
		-2	Do not estimate the partial derivatives of the optimal value function first. Conduct the sensitivity analysis with respect to all parameters.

A potential user of these sensitivity subroutines should be aware that the penalty function coded in SUMT-Version 4 does not have the

factor "1" in the quadratic loss term (see equation (1) in Section 2). Therefore, the expressions for the several gradients have an additional factor of "2" in the computer program which does not appear in the supporting theory of Section 2.

#### 4. New Computational Experience

In this Section, the four sample problems of Armacost and Fiacco (1974) are examined. Specifically, the convergence of the partial derivatives of the Lagrange multipliers is examined and the estimates of the gradient of the optimal value function obtained by the chain rule (equation (4)) and by the gradient of the Lagrangian (equation (3)) are compared. The problems are designated by the same letters as in the original paper.

Consider first a simple convex program. The problem is

$$\begin{aligned} \text{minimize} \quad f(x, \epsilon) &= \epsilon_1 + \epsilon_2 x_2 \\ \text{subject to} \quad g_1(x, \epsilon) &= \epsilon_1^2 - x_1^2 - x_2^2 \geq 0, \end{aligned}$$

for  $\epsilon_1 > 0$ . The analytical solution point and its gradient are given in Armacost and Fiacco (1974) as

$$x(\epsilon) = \begin{bmatrix} x_1(\epsilon) \\ x_2(\epsilon) \end{bmatrix} = \begin{bmatrix} -\frac{\epsilon_1}{\sqrt{1 + \epsilon_2^2}} \\ -\frac{\epsilon_1 \epsilon_2}{\sqrt{1 + \epsilon_2^2}} \end{bmatrix} .$$

$$f^*(\epsilon) = -\epsilon_1 \sqrt{1 + \epsilon_2^2} .$$

$$\nabla_{\epsilon} x(\epsilon) = \frac{1}{\sqrt{1 + \epsilon_2^2}} \begin{bmatrix} -1 & \frac{\epsilon_1 \epsilon_2}{(1 + \epsilon_2^2)} \\ -\epsilon_2 & \frac{-\epsilon_1}{(1 + \epsilon_2^2)} \end{bmatrix},$$

and

$$\nabla_{\epsilon} f^*(\epsilon) = (\partial f^*(\epsilon)/\partial \epsilon_1, \partial f^*(\epsilon)/\partial \epsilon_2)$$

$$= \left( -\sqrt{1 + \epsilon_2^2}, -\epsilon_1 \epsilon_2 / \sqrt{1 + \epsilon_2^2} \right).$$

The Lagrange multiplier and its gradient are analytically determined to be

$$u^*(\epsilon) = \sqrt{1 + \epsilon_2^2} / 2\epsilon_1,$$

and

$$\nabla_{\epsilon} u^*(\epsilon) = (-\sqrt{1 + \epsilon_2^2} / 2\epsilon_1^2, \epsilon_2 / (2\epsilon_1 \sqrt{1 + \epsilon_2^2})).$$

The numerical example had  $\epsilon_1 = 2$  and  $\epsilon_2 = 1$  yielding the following numerical results:

$$\begin{aligned} f^* &= -2\sqrt{2}, & \nabla_{\epsilon} f^* &= (-\sqrt{2}, -\sqrt{2}), \\ x^* &= \begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \end{bmatrix}, & \nabla_{\epsilon} x^* &= \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}, \\ u^* &= \sqrt{2}/4 & \nabla_{\epsilon} u^* &= (-\sqrt{2}/8, \sqrt{2}/8) \\ &\approx 0.353, & &\approx (-.177, .177). \end{aligned}$$

The numerical results obtained by the computer program are included in Table 2 for the optimal value function and Lagrange multiplier sensitivity. The values of the first order optimal value function sensitivity computed both by the chain rule (equation (4)) and by taking partial derivatives of the Lagrangian with respect to the parameters (equation (3))

TABLE 2  
TRAJECTORY RESULTS FOR PROBLEM B

Subproblem	f	Lagrangian		Chain rule		u	$\partial u / \partial \epsilon_1$	$\partial u / \partial \epsilon_2$
		$\partial f / \partial \epsilon_1$	$\partial f / \partial \epsilon_2$	$\partial f / \partial \epsilon_1$	$\partial f / \partial \epsilon_2$			
1	-1.9999	-1.9999	-.9999	-1.3333	-1.3333	.4999	-.3333	.1666
2	-2.5393	-1.5440	-1.2947	-1.4087	-1.4088	.3860	-.2100	.1761
3	-2.7765	-1.4439	-1.3833	-1.4139	-1.4139	.3609	-.1844	.1767
4	-2.8128	-1.4243	-1.4064	-1.4142	-1.4142	.3560	-.1790	.1768
5	-2.8245	-1.4127	-1.4123	-1.4142	-1.4142	.3531	-.1768	.1768
6	-2.8274	-1.4137	-1.4137	-1.4142	-1.4142	.3532	-.1767	.1768
7	-2.8282	-1.3899	-1.4141	-1.4142	-1.4142	.3475	-.1737	.1768
Analytical	-2.8282	-1.4142	-1.4142	-1.4142	-1.4142	.3537	-.1769	.1768

are presented in parallel. The results are also plotted in Figure 1 and portray the type of convergence experienced. While the previous results by Armacost and Fiacco (1974) clearly indicated a stability of the solution point and optimal value function and their gradients taken with respect to the parameters, Table 2 indicates that with the Lagrange multipliers, the same sort of stability is not found. It is well known that with barrier functions, the estimates of the Lagrange multipliers decrease in accuracy as the boundary is approached. The change in the value of u between subproblems 6 and 7 is an indication of this. It is no surprise, therefore, that the estimates of the gradient of the Lagrange multiplier behave in

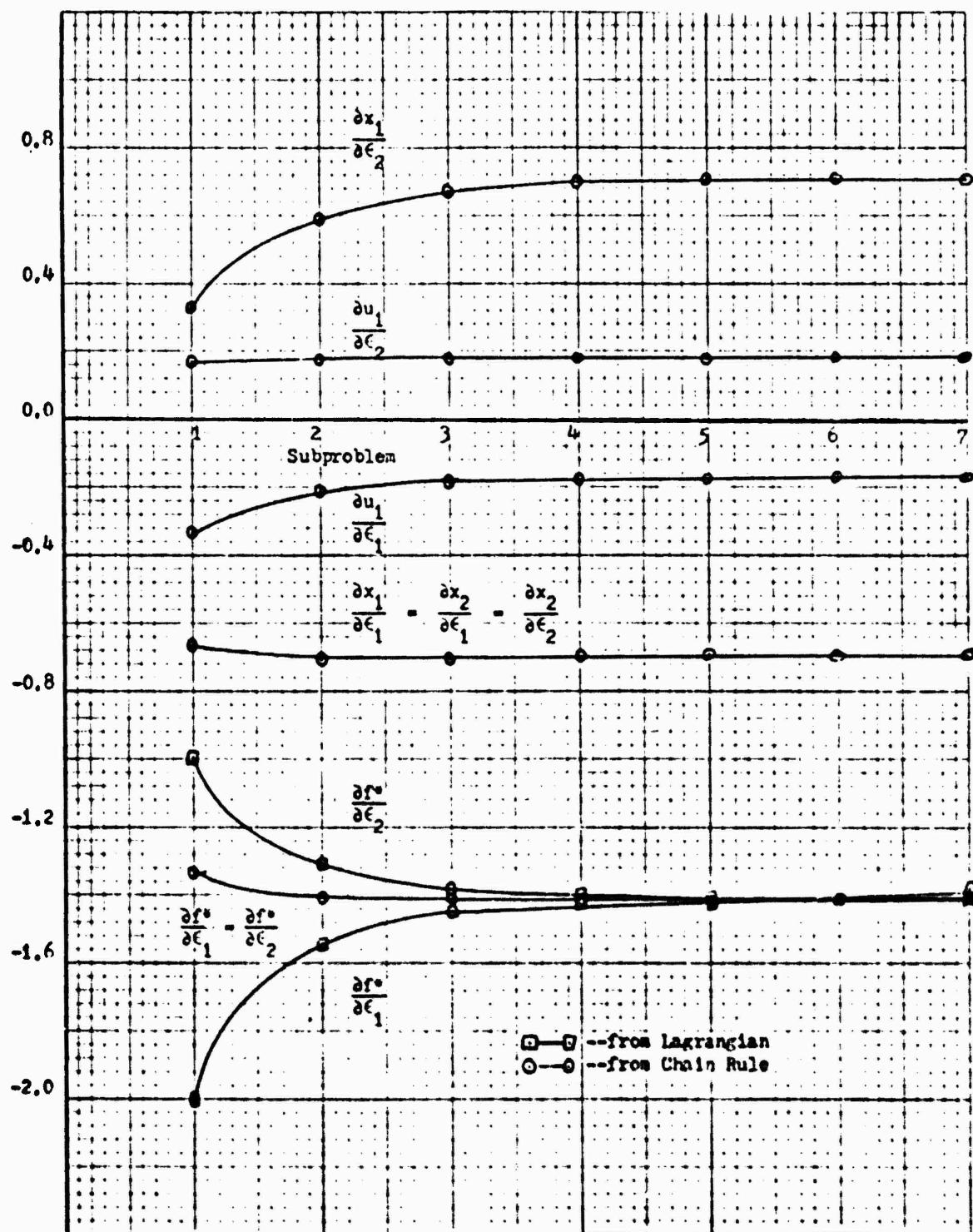


Fig. 1.--Partial derivatives for Problem 2.

a similar way. In addition, since the estimate of the gradient of the optimal value function obtained by evaluating the partial derivatives of the Lagrangian taken with respect to the parameters includes the estimates of the Lagrange multipliers, it too will not be as accurate an estimate as the boundary is approached.

The next problem considered is a nonconvex program with an equality constraint. The problem is to

$$\begin{aligned} \text{minimize} \quad f(x, \epsilon) &= x_1 + x_2 + \ln x_3 - x_4 \\ \text{subject to} \quad g_1(x, \epsilon) &= -x_1^2 + x_2 \geq 0, \\ g_2(x, \epsilon) &= x_1 \geq 0, \\ g_3(x, \epsilon) &= x_3 - \epsilon_1 \geq 0, \\ h_1(x, \epsilon) &= x_3^2 + x_4^2 - \epsilon_2^2 = 0, \end{aligned}$$

where  $\epsilon_2 \geq \epsilon_1 \geq 0$  and  $\epsilon_2 > 0$ .

The analytical solution is:

$$\begin{aligned} f^*(\epsilon) &= \ln \epsilon_1 - \sqrt{\epsilon_2^2 - \epsilon_1^2}, \\ x_1^*(\epsilon) &= x_2^*(\epsilon) = 0, \quad x_3^*(\epsilon) = 1, \quad x_4^*(\epsilon) = \sqrt{\epsilon_2^2 - \epsilon_1^2}, \\ u_1^*(\epsilon) &= u_2^*(\epsilon) = 1, \\ u_3^*(\epsilon) &= 1/\epsilon_1 + \epsilon_1/\sqrt{\epsilon_2^2 - \epsilon_1^2}, \\ \text{and} \quad w_1^*(\epsilon) &= 1/(2\sqrt{\epsilon_2^2 - \epsilon_1^2}). \end{aligned}$$

The numerical example used has  $\epsilon_1 = 1$  and  $\epsilon_2 = 2$ . The numerical solution derived analytically for the solution point, Lagrange multipliers and their gradients are:

$$\begin{aligned} f^* &\approx -1.732, \quad \nabla_{\epsilon} f^* \approx (1.578, -1.157), \\ x^* &\approx \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1.732 \end{bmatrix}, \quad \nabla_{\epsilon} x^* \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -0.577 & 1.154 \end{bmatrix}. \end{aligned}$$

$$u^* \approx \begin{bmatrix} 1 \\ 1 \\ 1.732 \end{bmatrix}, \quad \nabla_{\epsilon} u^* \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -.231 & -.385 \end{bmatrix},$$

$$w^* \approx .289, \quad \nabla_{\epsilon} w^* \approx (.0962, -.1925).$$

The computed results for a trajectory sensitivity analysis are shown in Table 3.

The partial derivatives of the Lagrange multipliers shown in Table 3 are the only ones which are non-zero. Note that at the fifth and sixth subproblems, the estimates of the gradients of  $u_3$  and  $w_1$  are reasonably close to the true values determined analytically. It is at that point also that the estimates of  $u_3$  and  $w_1$  are the closest to their true values at the solution point. Notice also that the estimate of  $\partial f^*/\partial \epsilon_2$  obtained from the partial derivative of the Lagrangian with respect to  $\epsilon_2$  is reasonably close to the true value. It is entirely dependent on  $w_1$  and as the estimate for  $w_1$  becomes less accurate, the error will be reflected in the first order sensitivity estimate of the optimal value function taken with respect to  $\epsilon_2$ . In the following example, the need for careful attention to the differencing increment is illustrated when the parameters are the right-hand sides of the constraints. For Problem C, a sensitivity analysis was conducted at the final subproblem for a range of differencing increments. The results were that the sensitivity estimates remained fairly constant over the range of differencing increments from  $10^{-7}$  to  $10^{-12}$ . Thus, the source of error in this case is solely the lack of accuracy of the estimates of the Lagrange multipliers for the binding constraints.

Two related problems called the Shell Primal and the Shell Dual were presented by Armacost and Fiacoo (1974). However, computational

TABLE 3  
TRAJECTORY RESULTS FOR PROBLEM C

subproblem	f	Lagrangian rule		Chain rule		$u_1$	$u_2$	$u_3$	$\partial u_2 / \partial \epsilon_1$	$\partial u_3 / \partial \epsilon_1$	$\partial u_1 / \partial \epsilon_2$	$\partial u_1 / \partial \epsilon_1$
		$\partial f / \partial \epsilon_1$	$\partial f / \partial \epsilon_2$	$\partial f / \partial \epsilon_1$	$\partial f / \partial \epsilon_2$							
1	.857	1.808	-1.500	1.252	-.948	.999	1.999	1.808	.375	.931	-.891	.222
2	-1.027	1.568	-1.216	1.538	-1.128	1.000	1.366	1.568	.304	.164	-.512	.128
3	-1.550	1.571	-1.172	1.573	-1.150	.999	1.123	1.570	.292	-.123	-.416	.104
4	-1.686	1.575	-1.158	1.576	-1.154	1.001	1.030	1.575	.289	-.203	-.392	.098
5	-1.720	1.574	-1.161	1.577	-1.154	.996	1.007	1.574	.290	-.219	-.388	.097
6	-1.729	1.576	-1.168	1.577	-1.154	.996	1.000	1.576	.292	-.219	-.390	.097
7	-1.731	1.585	-1.109	1.577	-1.154	1.000	1.006	1.585	.277	-.258	-.369	.092
8	-1.732	1.581	-1.045	1.577	-1.154	1.001	1.002	1.581	.261	-.303	-.348	.087
Analytical	-1.732	1.578	-1.157	1.578	-1.157	1.000	1.000	1.578	.289	-.231	-.385	.096

results were presented only for the Shell Dual. (The Shell Dual was developed as a test problem by the Shell Development Company and used by Colville (1968) in his comparative analysis of nonlinear programming codes.) Computational results are presented below for both the Shell Primal and the Shell Dual. The first problem considered is the Shell Primal.

$$\text{minimize } f(x, \epsilon) = \sum_{j=1}^n e_j x_j + \sum_{i=1}^n \sum_{j=1}^n x_i c_{ij} x_j + \sum_{j=1}^n d_j x_j^3$$

$$\text{subject to } g_i(x, \epsilon) = \sum_{j=1}^n a_{ij} x_j - \epsilon_i \geq 0, i=1, \dots, n,$$

with  $x_j \geq 0, j=1, \dots, n$ . The dual problem is much more difficult to solve and is the one most often used in computational comparisons. The Shell Dual is to

$$\text{maximize } f(x, \epsilon) = \sum_{j=1}^m \epsilon_j y_j - \sum_{i=1}^n \sum_{j=1}^n x_i c_{ij} x_j - 2 \sum_{i=1}^n d_i x_i^3$$

$$\text{subject to } g_i(x, \epsilon) = e_i + 2 \sum_{j=1}^n c_{ji} x_j + 3d_i x_i^2 - \sum_{j=1}^m a_{ji} y_j \geq 0, i=1, \dots, n,$$

with  $x_i \geq 0, i=1, \dots, n$ , and  $y_j \geq 0, j=1, \dots, m$ . In the numerical example used here,  $n = 5$  and  $m = 10$ . The problem data is given in Table 3 of Armacost and Fiacco (1974) and in Appendix C. The parameters of the sensitivity analysis are the variables  $\epsilon_i, i=1, \dots, 10$ , the components of the right-hand sides of the primal constraints.

As a brief aside, the computational solution of the Shell Dual provided the motivation for some of the recent work in parametric sensitivity analysis by Armacost and Fiacco. Specifically, Armacost and Fiacco (1974) noted that in solving the dual problem, the partial

derivatives of the dual variables with respect to the right-hand sides of the primal constraints were obtained. With the correspondence between the dual variables and Lagrange multipliers and their interpretation as the partial derivatives of the optimal value function with respect to the right-hand sides of the primal constraints, it appeared that the second order partial derivatives of the optimal value function had been obtained. The calculations supported this conjecture since the matrix of partial derivatives of the dual variables with respect to the parameters was symmetric. Armacost and Fiacco (1976) have shown that when the parameters are the components of the right-hand side only, then the gradient of the Lagrange multiplier vector taken with respect to the parameters is the Hessian of the optimal value function. This matrix will be computed using the Shell Primal and then compared with the Hessian obtained by solving the Shell Dual.

In solving all of the sample problems, the option to screen the sensitivity estimates was used resulting in the partial derivatives being computed only for those parameters which affected the optimal value function by more than 0.1 percent of its current value. Annotated computer output for the final subproblem with sensitivity analysis data for the Shell Primal is shown in Figure 2. (The annotation applies to the computer output in Figures 3, 4, 5 and 6 as well.) The parameters are represented by the letter "A" vice "f" in the computer output. Similar output for the Shell Dual is shown in Figure 7. (Compare the sensitivity analysis portion with Figure 4 of Armacost and Fiacco (1974).) The sensitivity estimates for both problems were obtained by conducting a trajectory sensitivity analysis, i.e., a sensitivity analysis performed at each subproblem along the minimizing trajectory. Since the Shell Dual is a maximization problem and SINT is coded to solve a minimization problem, problem 6 is solved

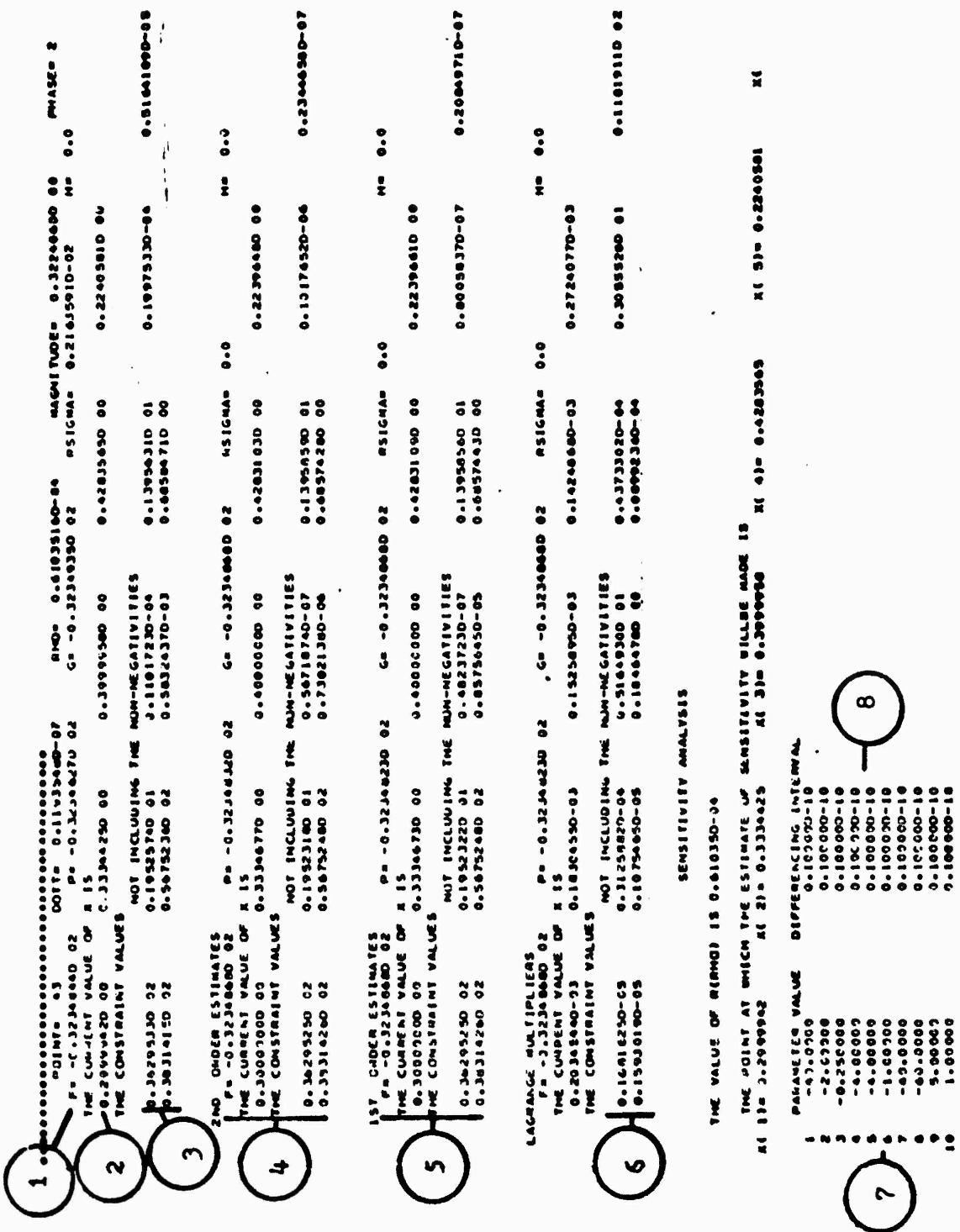


Fig. 2.--Shell Primal subproblem and sensitivity output.

## OPTIONAL VALUE FUNCTION SENSITIVITY

- 9 DF/DAT 11= 2.16912705-05 CF/DAT 21= 0.4172540-04 DF/DAT 31= 3.166927 DF/DAT 41= 0.1073590-05 DF/DAT 51= 0.1062560-05 DF/DAT 61= 0.01073590-05 DF/DAT 71= 0.1169230-05 DF/DAT 81= 0.01062560-05 DF/DAT 91= 0.00992230-05 DF/DAT 101= 0.00992230-05
- Detailed sensitivity results follow from parameters 9, 10, 11, 12, 13.
- 10
- $\frac{d}{d\lambda}$ -derivatives are with respect to parameter 9.  
Out 11= 0.97337410-01 Out 21= 0.97337410-01 Out 31= 0.1969975 Out 41= 0.2469346 Out 51= 0.47996740-01 Out 61= 0.47996740-01 Out 71= 0.11693790-01 Out 81= 0.1099914 Out 91= 0.1099914 Out 101= 0.1099914
- $\frac{d^2}{d\lambda^2}$ -derivatives with respect to parameter 9.  
Out 11= 2.1157470-05 Out 21= 0.36630320-04 Out 31= 0.36630320-04 Out 41= 0.36630320-04 Out 51= 0.36630320-04 Out 61= 0.36630320-04 Out 71= 0.11693790-01 Out 81= 0.1099914 Out 91= 0.1099914 Out 101= 0.1099914
- $\frac{d^3}{d\lambda^3}$ -derivatives are with respect to parameter 9.  
Out 11= 0.5727610-16 Out 21= 0.1471101 Out 31= 0.35330800-06 Out 41= 0.49171610-01 Out 51= 0.98193170-01 Out 61= 0.98193170-01 Out 71= 0.46774520-17 Out 81= 0.7536470-05 Out 91= 0.3662008 Out 101= 0.12307150-04 Out 11= 0.5051511
- $\frac{d^4}{d\lambda^4}$ -derivatives with respect to parameter 9.  
Out 11= 0.1556120-39 Out 21= 0.55791140-08 Out 31= 0.23921210-01 Out 41= 0.23921210-01 Out 51= 0.12326000-04 Out 61= 0.12326000-04 Out 71= 0.46774520-18 Out 81= 0.7536470-06 Out 91= 0.3662008 Out 101= 0.12326000-04 Out 11= 0.5051511
- 11
- $\frac{d}{d\lambda}$ -derivatives are with respect to parameter 10.  
Out 11= 0.9727610-16 Out 21= 0.1471101 Out 31= 0.35330800-06 Out 41= 0.49171610-01 Out 51= 0.98193170-01 Out 61= 0.98193170-01 Out 71= 0.46774520-17 Out 81= 0.7536470-05 Out 91= 0.3662008 Out 101= 0.12307150-04 Out 11= 0.5051511
- $\frac{d^2}{d\lambda^2}$ -derivatives with respect to parameter 10.  
Out 11= 2.1157470-05 Out 21= 0.36630320-04 Out 31= 0.36630320-04 Out 41= 0.36630320-04 Out 51= 0.36630320-04 Out 61= 0.36630320-04 Out 71= 0.11693790-01 Out 81= 0.1099914 Out 91= 0.1099914 Out 101= 0.1099914
- $\frac{d^3}{d\lambda^3}$ -derivatives are with respect to parameter 10.  
Out 11= 0.5727610-16 Out 21= 0.1471101 Out 31= 0.35330800-06 Out 41= 0.49171610-01 Out 51= 0.98193170-01 Out 61= 0.98193170-01 Out 71= 0.46774520-17 Out 81= 0.7536470-05 Out 91= 0.3662008 Out 101= 0.12307150-04 Out 11= 0.5051511
- $\frac{d^4}{d\lambda^4}$ -derivatives with respect to parameter 10.  
Out 11= 0.1556120-39 Out 21= 0.55791140-08 Out 31= 0.23921210-01 Out 41= 0.23921210-01 Out 51= 0.12326000-04 Out 61= 0.12326000-04 Out 71= 0.46774520-18 Out 81= 0.7536470-06 Out 91= 0.3662008 Out 101= 0.12326000-04 Out 11= 0.5051511
- 12
- $\frac{d}{d\lambda}$ -derivatives are with respect to parameter 11.  
Out 11= 0.9643100-01 Out 21= 0.9643100-01 Out 31= 0.3669971 Out 41= 0.2201358 Out 51= 0.2795040-01 Out 61= 0.2795040-01 Out 71= 0.1092930-06 Out 81= 0.1092930-06 Out 91= 0.1092930-06 Out 101= 0.1092930-06
- $\frac{d^2}{d\lambda^2}$ -derivatives with respect to parameter 11.  
Out 11= 2.1677480-06 Out 21= 0.66766110-06 Out 31= 3.471577 Out 41= 0.35286940-04 Out 51= 0.615989 Out 61= 0.29583760-04 Out 71= 0.8202930-06 Out 81= 0.81092490-07 Out 91= 1.589066 Out 101= 1.589066
- $\frac{d^3}{d\lambda^3}$ -derivatives are with respect to parameter 11.  
Out 11= 0.5727610-16 Out 21= 0.1471101 Out 31= 0.35330800-06 Out 41= 0.49171610-01 Out 51= 0.98193170-01 Out 61= 0.98193170-01 Out 71= 0.46774520-17 Out 81= 0.7536470-05 Out 91= 0.3662008 Out 101= 0.12307150-04 Out 11= 0.5051511
- $\frac{d^4}{d\lambda^4}$ -derivatives with respect to parameter 11.  
Out 11= 0.1556120-39 Out 21= 0.55791140-08 Out 31= 0.23921210-01 Out 41= 0.23921210-01 Out 51= 0.12326000-04 Out 61= 0.12326000-04 Out 71= 0.46774520-18 Out 81= 0.7536470-06 Out 91= 0.3662008 Out 101= 0.12326000-04 Out 11= 0.5051511
- 13
- $\frac{d}{d\lambda}$ -derivatives are with respect to parameter 12.  
Out 11= 0.3127272D-01 Out 21= 0.39570130-01 Out 31= 0.32994070-04 Out 41= 0.73587080-01 Out 51= 0.1542692 Out 61= 0.1542692
- $\frac{d^2}{d\lambda^2}$ -derivatives with respect to parameter 12.  
Out 11= 0.1556120-39 Out 21= 0.66691631-05 Out 31= 0.61699777 Out 41= 0.11553320-04 Out 51= 0.24716170-04 Out 61= 0.24716170-04 Out 71= 0.79193590-09 Out 81= 0.42453470-08 Out 91= 0.6347798 Out 101= 0.1020008

Fig. 2.--Continued.

<u>Identifier</u>	<u>Annotation</u>
Identifier	Meaning
1	$F =$ the value of the objective function at the current solution point.
2	The value of the components of the solution point of the current subproblem, here, $x_1, \dots, x_5$ .
3	The value of the constraints evaluated at the current solution point, i.e., $\epsilon_1, \dots, \epsilon_{10}$ .
4	The data corresponding to 1 - 3 above when the solution point is a second order estimate based on the values in 2.
5	The same as 4, but the solution point is a first order estimate based on the values in 2. These values and those in 4 are extrapolations from the current solution estimate.
6	The estimates of the Lagrange multipliers based on the current value of $r$ (RHO) and the current estimates of the solution point, in this case as $u_i = r/\epsilon_i(x, \epsilon)$ , $i=1, \dots, 10$ .
7	The value of the parameters, $\epsilon_i$ , $i=1, \dots, 10$ .
8	The value of the differencing increment used in the central differencing formula for each of the parameters.
9	The estimates of the gradient of the optimal value function calculated by equation (3).
10	The parameters whose associated partial derivatives affect the optimal value function by more than 0.001 of its current value.
11	The first order sensitivity of the solution point calculated by equation (2).
12	The first order sensitivity of the Lagrange multipliers calculated using the method described following equation (2).
13	The partial derivative of the optimal value function taken with respect to the indicated parameter and calculated by equation (4).

FIG. 7.--Continued.

PHASE = 2

P11 411 04 001112570-07 R110 0.610316D-06 MAGNITUDE = 0.56436170 00  
 P2 0.323491D 02 P3 0.323491D 02 R111 0.32110130 02 H110 0.7034342D-02  
 THE CURRENT VALUE OF X15 C= 0.323491D 02 H111 0.381135D-04  
 0.164426D-05 0.3131440D-04 0.51742730 01 0.381135D-04  
 0.153786D-05 0.1073433-C5 0.1043153D 00 0.381135D-04  
 0.323491D 00 0.42315400 00 0.22403110 00 0.3000026D 00  
 THE CONSTRAINT VALUES NOT INCLUDING THE NON-NEGATIVITIES  
 0.2138172D-03 0.1526649D-03 0.1427815D-03 0.2729410D-03

2ND ORDER ESTIMATES

P11 0.323491D 02 P2 0.323491D 02 C= 0.323491D 02 ASIGMA= 0.0 H110 0.0  
 THE CURRENT VALUE OF X15  
 0.3234915D-08 0.713086D-07 J= 0.51742730 01 0.1073435D-06  
 0.376172D-08 0.5906753-08 0.10381100 00 0.20400000D-06  
 C.3994946D 00 0.4283C773 00 0.223396C2D 00 0.30000000 00  
 THE CONSTRAINT VALUES NOT INCLUDING THE NON-NEGATIVITIES  
 0.4843248D-06 0.3767802D-06 0.333988780-06 0.10856370-06 0.32398641D-06

LAGRANGE MULTIPLIERS

P11 0.323491D 02 P2 0.323491D 02 C= 0.323491D 02 ASIGMA= 0.0 H110 0.0  
 THE CURRENT VALUE OF X15  
 0.3623720 02 0.10491071 01 0.117589D-06 0.1392113D 01 0.19931698D-04  
 0.3823510 02 0.36652423 02 0.5993344D-03 0.6666339D-03 0.2034486D-03  
 0.1523665D-03 0.14226867D-03 0.2226017D-03 0.1830437D-03  
 THE CONSTRAINT VALUES NOT INCLUDING THE NON-NEGATIVITIES  
 0.29946020 00 0.33282600 00 0.30927320 00 0.42747200 00 0.2236204D 00

SENSITIVITY ANALYSIS

THE VALUE OF R110 IS 0.610316D-06

THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY WILL BE MADE IS  
 X(1)= 0.164426D-05 X(2)= 0.3131440D-04 X(3)= 0.5174273  
 X(4)= 0.1073433-C5 X(5)= 0.1043153D-05 X(6)= 0.381135D-04 X(7)= 0.381135D-04  
 R111= 0.153786D-05 X(8)= 0.1377480U-05 X(9)= 0.10381100 X(10)= 0.3811351D-04 X(11)= 0.3000026  
 R112= 0.323491D 00 X(12)= 0.4283C773 X(13)= 0.22339631 X(14)= 0.33398858

PARAMETER VALUE DIFFERENCING INTERVAL

1 -4C.323491D 0-10  
 2 -2.323491D 0-10 C.10000D-10  
 3 -0.250326D 0-10 0.10000D-10  
 4 -0.033350 0-10 0.033350-10  
 5 -4.000000 0-10 0.10000D-10  
 6 -1.260000 0-10 0.10000D-10  
 7 -0.033000 0-10 0.10000D-10  
 8 -0.000000 0-10 0.10000D-10  
 9 2.000000 0-10 0.10000D-10  
 10 1.000000 0-10 0.10000D-10

Fig. 3. --Shell Dual subproblem and sensitivity output.

## OPTIMAL VALUE FUNCTION SENSITIVITY

$\frac{\partial f}{\partial DAI}$  J1= 0.3       $\frac{\partial f}{\partial DAI}$  J1= 0.0  
 $\frac{\partial f}{\partial DAI}$  J1= -11.84066       $\frac{\partial f}{\partial DAI}$  J1= 0.0  
 $\frac{\partial f}{\partial DAI}$  J1= 0.0

## DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS

J = 5 = 6 = 9 .

x-derivatives are with respect to parameter • 3  
DAI 1= -2.311903220-06 DAI 2= 0.-397460500-05 DAI 3= 4.-066262  
DAI 7= 0.-0.37261930-36 DAI 8= 0.-1693743 DAI 9= 0.3703780-04 DAI 11= 0.3993361  
DAI 13= -0.1986140 DAI 14= 0.28526035 DAI 15= -0.677791160-01 DAI

u-derivatives with respect to parameter 3  
DAI 1= 11.-0.3999956 DAI 2= 0.-97368610-01 DAI 3= -0.1999973 DAI 4= 0.2860099  
DAI 5= 0.67070100-01 DAI 6= 0.2860099  
DAI 7= 0.67070100-01 DAI 8= 0.2860099  
DAI 9= 0.2860099  
DAI 10= 0.2860099  
DAI 11= 0.2860099  
DAI 12= 0.2860099  
DAI 13= 0.2860099  
DAI 14= 0.2860099  
DAI 15= 0.2860099

Optimality/DAI = -9.17412

Optimality/DAI = -3.06121

Optimality/DAI = -11.83397

x-derivatives are with respect to parameter • 5  
DAI 1= 0.1596470-037 DAI 2= 0.-73654260-03 DAI 3= -0.-5632817 DAI 4= -0.12354650-06 DAI 5= 0.-3043776 DAI 6= 0.0151008  
DAI 7= -0.0.250250-06 DAI 8= -0.500030250-06 DAI 9= 0.2371030-01 DAI 10= 0.12779510-06 DAI 11= 0.97746010-05 DAI 12= 0.-1463753  
DAI 13= -0.1034967-05 DAI 14= -0.4909590-01 DAI 15= 0.97901440-01 DAI

u-derivatives with respect to parameter 0

DAI 1= -0.56912250-06 DAI 2= 0.-1071216 DAI 3= -0.3519960-06 DAI 4= -0.4919960-01 DAI 5= 0.98181980-01 DAI  
DAI 6= 0.98181980-01 DAI 7= 0.98181980-01 DAI 8= 0.98181980-01 DAI 9= 0.98181980-01 DAI 10= 0.98181980-01 DAI 11= 0.98181980-01 DAI 12= 0.98181980-01 DAI 13= 0.98181980-01 DAI 14= 0.98181980-01 DAI 15= 0.98181980-01 DAI

Optimality/DAI = -3.06121

Optimality/DAI = -11.83397

x-derivatives are with respect to parameter • 6  
DAI 1= -0.16418230-06 DAI 2= -0.62724270-06 DAI 3= 3.-463049 DAI 4= 0.35522310-06 DAI 5= -0.151012 DAI 6= 7.103666  
DAI 7= -0.293229210-06 DAI 8= -0.1096460-07 DAI 9= 1.-563562 DAI 10= 0.23973670-06 DAI 11= 0.1994387 DAI 12= 0.-0.303620-01 DAI  
DAI 13= -0.3439777 DAI 14= 0.2976306 DAI 15= 0.28173220-01 DAI

u-derivatives with respect to parameter 6

DAI 1= -0.16456450-01 DAI 2= 0.-0.9499971 DAI 3= -0.3499971 DAI 4= 0.2281993 DAI 5= 0.27081650-01 DAI  
DAI 6= 0.27081650-01 DAI 7= 0.27081650-01 DAI 8= 0.27081650-01 DAI 9= 0.27081650-01 DAI 10= 0.27081650-01 DAI 11= 0.27081650-01 DAI 12= 0.27081650-01 DAI 13= 0.27081650-01 DAI 14= 0.27081650-01 DAI 15= 0.27081650-01 DAI

x-derivatives are with respect to parameter • 9  
DAI 1= 0.15591490-09 DAI 2= -0.6715160-05 DAI 3= 0.-1607744 DAI 4= 0.11894380-06 DAI 5= 0.-23771290-01 DAI 6= 1.-980583  
DAI 7= 0.-79480610-06 DAI 8= 0.-0.42114120-06 DAI 9= 0.-0.301746 DAI 10= 0.-0.24863862-06 DAI 11= 0.-0.23057130-04 DAI 12= 0.-0.23465662-01 DAI  
DAI 13= -0.26938610-06 DAI 14= 0.73684600-01 DAI 15= 0.1501222 DAI

u-derivatives with respect to parameter 9  
DAI 1= 6.2110650-06 DAI 2= -0.39370250-01 DAI 3= 0.32669930-06 DAI 4= 0.73793260-01 DAI 5= 0.1542566 DAI  
DAI 6= 0.1542566 DAI 7= 0.1542566 DAI 8= 0.1542566 DAI 9= 0.1542566 DAI 10= 0.1542566 DAI 11= 0.1542566 DAI 12= 0.1542566 DAI 13= 0.1542566 DAI 14= 0.1542566 DAI 15= 0.1542566 DAI

Fig. 3.—Continued.

by minimizing the negative of the objective function and therefore, the results in Fig. 3 for the objective function value and the values of its partial derivatives must be multiplied by -1 to obtain the correct results. The variables  $X(1) - X(10)$  in the Shell Dual correspond to the Lagrange multipliers in the Shell Primal and the variables  $X(11) - X(15)$  in the Dual correspond to  $X(1) - X(5)$  in the Primal. The components of the Hessian of the optimal value function then are the partial derivatives of  $X(1) - X(10)$  in the Shell Dual and the u-derivatives in the Shell Primal. The dual variables, Lagrange multipliers, the partial derivatives of the optimal value function obtained by means of the Lagrangian, and those obtained using the chain rule are compared in Table 4.

TABLE 4  
FIRST ORDER SENSITIVITY COMPARISON

i	Shell Primal			Shell Dual		
	$u_i$	$\partial f / \partial \epsilon_i$ (Lag.)	$\partial f / \partial \epsilon_i$ (C.R.)	$x_i$	$\partial f / \partial \epsilon_i$ (Lag.)	$\partial f / \partial \epsilon_i$ (C.R.)
1	$.168 \times 10^{-5}$	$.168 \times 10^{-5}$	-	$.168 \times 10^{-5}$	0.0	-
2	$.312 \times 10^{-4}$	$.312 \times 10^{-4}$	-	$.313 \times 10^{-4}$	0.0	-
3	5.1649	5.1649	5.1740	5.1742	5.1741	5.1741
4	$.437 \times 10^{-4}$	$.437 \times 10^{-4}$	-	$.438 \times 10^{-4}$	0.0	-
5	3.0555	3.0554	3.0610	3.0610	3.0610	3.0612
6	11.8191	11.8190	11.8395	11.8405	11.8406	11.8397
7	$.159 \times 10^{-5}$	$.159 \times 10^{-5}$	-	$.159 \times 10^{-5}$	0.0	-
8	$.107 \times 10^{-5}$	$.107 \times 10^{-5}$	-	$.107 \times 10^{-5}$	0.0	-
9	.1046	.1046	.1038	.1043	.1044	.1040
10	$.889 \times 10^{-4}$	$.889 \times 10^{-4}$	-	$.891 \times 10^{-4}$	$.177 \times 10^{-3}$	-

Since the preliminary screening option was used, detailed sensitivity estimates for the parameters which correspond to the non-binding inequality

constraints were not computed in Figures 2 and 3. In the Shell Primal, the Lagrange multipliers and the partial derivatives obtained by means of the Lagrangian correspond exactly. As noted in the discussions following Problems B and C, the estimates of the Lagrange multipliers for the binding constraints are very sensitive near the boundary of the constraint set. This explains the slight variation between these estimates and the other sensitivity estimates shown in Table 4 for the binding Primal constraints.

Now compare the second order sensitivity estimates with respect to the parameters which are the right-hand sides of the binding Primal constraints. Let  $H_D$  and  $H_P$  denote the submatrices of the Hessian of the optimal value function for the Dual and Primal respectively, obtained by deleting the rows and columns corresponding to the non-binding Primal inequality constraints. Thus, the components of  $H_D$  are  $\partial x_i / \partial \epsilon_j$ ,  $i, j = 3, 5, 6, 9$ , and the components of  $H_P$  are  $\partial u_i / \partial \epsilon_j$ ,  $i, j = 3, 5, 6, 9$ . From the computer output, these matrices are:

$$H_D = \begin{bmatrix} 4.0642 & -.5653 & 3.4650 & .1894 \\ -.5653 & .5043 & -.6151 & .0237 \\ 3.4650 & -.6151 & 7.1850 & 1.5885 \\ .1894 & .0237 & 1.5885 & .8343 \end{bmatrix},$$

$$\text{and } H_P = \begin{bmatrix} 4.0710 & -.5663 & 3.4715 & .1899 \\ -.5662 & .5051 & -.6158 & .0239 \\ 3.4715 & -.6158 & 7.1929 & 1.5890 \\ .1899 & .0239 & 1.5890 & .8347 \end{bmatrix}.$$

Both  $H_D$  and  $H_P$  are symmetric, and while the agreement is not exact, it is very close as anticipated.

Armacost and Fiacco (1974) indicate that the differencing interval

used in the sensitivity analysis can affect the accuracy of the results. For this reason, an option was provided by Armacost and Mylander (1973) to conduct the sensitivity analysis at the final subproblem for a range of differencing intervals. It is even more important to be cautious when dealing with right-hand side perturbations as the following discussion indicates. The results shown above for the Shell Primal and the Shell Dual were obtained with a trajectory sensitivity analysis and at the final subproblem, the differencing increment used in the central differencing formulas was  $10^{-11}$ . The Hessian submatrices  $H_p$  and  $H_D$  were found to be very close. When the problems were solved using a sensitivity analysis at the final subproblem only with a differencing increment of  $10^{-9}$ , the diagonal elements of  $H_p$  were considerably different from those of  $H_D$ . The problems were solved again with a sensitivity analysis performed at the final subproblem for a range of values of the differencing increment, ranging from  $10^{-6}$  to  $10^{-11}$ . The components of  $H_D$  remained constant as did the non-diagonal elements of  $H_p$  which were equal to the non-diagonal elements of  $H_D$ . The diagonal elements of  $H_p$  did not remain constant and their variation is depicted in Table 5.

The final example considered in this section is called the cattle feed problem. It was formulated and originally presented by van de Panne and Popp (1963). Armacost and Fiacco (1974) presented the cattle feed problem to illustrate an application of the sensitivity analysis. The additional sensitivity results are presented here for completeness. The problem is a chance-constrained program to determine the mix of inputs to cattle feed that will satisfy nutritive constraints and minimize the cost of the cattle feed. The protein content of the components is a random

TABLE 5  
VARIATION IN THE COMPONENTS OF  $H_p$

$\Delta$	$\frac{\partial^2 f}{\partial \epsilon_i^2}$			
	i=3	i=5	i=6	i=9
$10^{-6}$	-3148.3	-383.81	-89157.9	.8342
$10^{-7}$	-27.228	-3.328	-851.345	.8347
$10^{-8}$	3.758	.466	-1.389	.8347
$10^{-9}$	4.067	.504	7.107	.8347
$10^{-10}$	4.071	.505	7.191	.8347
$10^{-11}$	4.071	.505	7.192	.8347

variable, normally distributed with a mean and variance determined experimentally. The application of the sensitivity analysis included the standard deviations as parameters with the interpretation that if the solution were sensitive to a standard deviation, more sampling would be indicated in order to obtain a sharper estimate. The statement of the problem is

$$\text{minimize } f(x, \epsilon) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$

$$\text{subject to } g_1(x, \epsilon) = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 - \epsilon_7 \geq 0,$$

$$g_2(x, \epsilon) = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4 \quad F$$

$$+ \epsilon_5 \sqrt{\epsilon_1^2 x_1^2 + \epsilon_2^2 x_2^2 + \epsilon_3^2 x_3^2 + \epsilon_4^2 x_4^2} - \epsilon_6 \geq 0,$$

$$h_1(x, \epsilon) = x_1 + x_2 + x_3 + x_4 - 1 = 0,$$

with  $x_i \geq 0$ ,  $i=1,2,3,4$ . The notation is slightly different from Armacost and Fiacco (1974). Here, the parameters in the sensitivity analysis are denoted  $\epsilon_i$ . The correspondence with the previous work is  $\epsilon_i = \sigma_i$ ,  $i=1,2,3,4$ .

$\epsilon_5 = \phi$ ,  $\epsilon_6 = p_m$  and  $\epsilon_7 = o_m$ . The problem data from Table 6 of Armacost and Fiacco (1974) are included in Appendix C. The sensitivity results for Problem F are shown in Figure 4. Again, the preliminary screening option was used to avoid calculating sensitivity estimates which did not affect the optimal value function significantly. Here, the components of the gradient of the optimal value function obtained directly from the gradient of the Lagrangian and the estimate of the gradient obtained by application of the chain rule are very close. The values of the Lagrange multipliers estimated in the SUMT program are  $u_1 = 0.58037$ ,  $u_2 = -0.41005$  and  $w_1 = -18.3738$ . The multipliers  $u_1$  and  $u_2$  correspond to  $\partial f^*/\partial \epsilon_7$  and  $\partial f^*/\partial \epsilon_6$  respectively. Also note that second order sensitivity information for  $\epsilon_6$  and  $\epsilon_7$  is available since they are the right-hand sides of the two inequality constraints. Namely,

$$\begin{aligned}\partial^2 f^*/\partial \epsilon_7^2 &= \partial u_1/\partial \epsilon_7 = .0112929, \\ \partial^2 f^*/\partial \epsilon_6 \partial \epsilon_7 &= \partial u_1/\partial \epsilon_6 = .45706 \times 10^{-6}, \\ \partial^2 f^*/\partial \epsilon_7 \partial \epsilon_6 &= \partial u_2/\partial \epsilon_7 = .45671 \times 10^{-6}, \\ \text{and } \partial^2 f^*/\partial \epsilon_6^2 &= \partial u_2/\partial \epsilon_6 = .46212 \times 10^{-3}.\end{aligned}$$

The results shown in Figure 4 were obtained using a sensitivity analysis at the final subproblem for a range of differencing increments. For other values of the differencing increments not shown here, the optimal value function sensitivity estimates with respect to  $\epsilon_6$  and  $\epsilon_7$  obtained directly from the gradient of the Lagrangian and the cross-partials of  $u_1$  and  $u_2$  with respect to  $\epsilon_6$  and  $\epsilon_7$  do vary somewhat while the other sensitivity estimates remain relatively constant. In a practical sense, however, the cross-partials of  $u_1$  and  $u_2$  with respect to  $\epsilon_6$  and  $\epsilon_7$  are constant since they are of the order of  $10^{-6}$ .

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## SENSITIVITY ANALYSIS

THE VALUE OF PERIOD IS 0.070590-03

THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY BILLET NAME IS  
DF/DAC 110 0.3107523 DF/DAC 210 0.31321760-03 DF/DAC 310 0.3120959 DF/DAC 410 0.31066740-03 DF/DAC 510 0.31067040-03 DF/DAC 610 0.31067040-03 DF/DAC 710 0.31067040-03

PARAMETER	VALUE	DIFFERENCING INTERVAL
1	3.937000	0.100000-03
2	7.000000	0.100000-03
3	6.977000	0.100000-03
4	6.787000	0.100000-03
5	1.000000	0.100000-03
6	21.57700	0.100000-03
7	9.03000	0.100000-03

## OPTIMAL VALUE FUNCTION SENSITIVITY

DF/DAC 110 0.31067040-03 DF/DAC 210 0.31321760-03 DF/DAC 310 0.3120959 DF/DAC 410 0.31066740-03 DF/DAC 510 0.31067040-03 DF/DAC 610 0.31067040-03 DF/DAC 710 0.31067040-03

DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS  
1 + 2 + 3 + 4 + 5 + 6 + 7+ DERIVATIVES ARE WITH RESPECT TO PARAMETER 1  
DF/DAC 110=0.59312740-03 DF/DAC 210=0.23347940-02 DF/DAC 310=-0.13001700-03 DF/DAC 410=0.50731050-02 DF/DAC- DERIVATIVES WITH RESPECT TO PARAMETER 2  
DF/DAC 110=-0.44114010-01 DF/DAC 210=-0.94776520-02 DF/DAC+ DERIVATIVES WITH RESPECT TO PARAMETER 3  
DF/DAC 110=-0.6374103 DF/DACDF/DAC 110/200 = 0.99994820-03  
oooooooooooo+ DERIVATIVES ARE WITH RESPECT TO PARAMETER 4  
DF/DAC 110=-0.10178700-01 DF/DAC 210=0.35000010-02 DF/DAC 310=0.28897700-04 DF/DAC 410=0.120236000-03 DF/DAC- DERIVATIVES WITH RESPECT TO PARAMETER 5  
DF/DAC 110=0.70720770-01 DF/DAC 210=0.28866270-02 DF/DAC+ DERIVATIVES WITH RESPECT TO PARAMETER 6  
DF/DAC 110=0.2101953 DF/DACDF/DAC 110/200 = 0.200173  
oooooooooooo+ DERIVATIVES ARE WITH RESPECT TO PARAMETER 7  
DF/DAC 110=-0.24527100-01 DF/DAC 210=0.10000030-02 DF/DAC 310=0.000007370-04 DF/DAC 410=0.30007400-01 DF/DAC- DERIVATIVES WITH RESPECT TO PARAMETER 8  
DF/DAC 110=0.10000031 DF/DAC 210=0.000007370-02 DF/DAC+ DERIVATIVES WITH RESPECT TO PARAMETER 9  
DF/DAC 110=0.00037270 DF/DACDF/DAC 110/200 = 0.00037270  
oooooooooooo+ DERIVATIVES ARE WITH RESPECT TO PARAMETER 10  
DF/DAC 110=-0.23501210-01 DF/DAC 210=-0.61791270-02 DF/DAC 310=0.300783270-02 DF/DAC 410=0.033280700-01 DF/DAC- DERIVATIVES WITH RESPECT TO PARAMETER 11  
DF/DAC 110=0.10000031 DF/DAC 210=0.000007370-02 DF/DAC+ DERIVATIVES WITH RESPECT TO PARAMETER 12  
DF/DAC 110=0.00071110-02 DF/DACDF/DAC 110/200 = 0.0119620  
oooooooooooo+ DERIVATIVES ARE WITH RESPECT TO PARAMETER 13  
DF/DAC 110=0.00000000-01 DF/DAC 210=0.00000000-02 DF/DAC 310=0.10007200 DF/DAC 410=0.00000000-01 DF/DAC- DERIVATIVES WITH RESPECT TO PARAMETER 14  
DF/DAC 110=0.00000000-01 DF/DAC 210=0.00000000-02 DF/DAC+ DERIVATIVES WITH RESPECT TO PARAMETER 15  
DF/DAC 110=0.00000000-01 DF/DACDF/DAC 110/200 = 0.00000000  
oooooooooooo

Fig. 4.--Sensitivity results for Problem F.

##### 5. Large-Scale, Multi-item Inventory Model

Traditionally, inventory models have been formulated to minimize some function of the ordering, holding and shortage (or backorder) costs subject to various constraints. Schrady and Choe (1971) have formulated an inventory model which appears to have much greater relevance for an inventory system such as the U. S. Naval supply system. For the Navy, the costs used in the traditional models may be quite artificial while the real objective of the system is to maximize the service to the Fleet, an objective equivalent to minimization of stockouts. In addition, the stock points of the Naval supply system have investment and reorder workload constraints that are real and binding. Schrady and Choe consider a multi-item inventory system with the specific objective function of minimizing the total time-weighted shortages. The decision variables are the "reorder quantities" and the "reorder points," the decisions of how much to order and when to order, for each item in the inventory. Clearly, it is of interest to know "if" and "by how much" these variables and the value of the objective function will change if certain parameters change. Schrady and Choe solved a small example problem using the SUMT computer code. McCormick (1972) has shown how the special structure of this inventory model can be used to facilitate the use of the SUMT code to solve very large inventory problems. Here, the sensitivity analysis is applied to this model, a large-scale inventory system, and it is illustrated by means of the example used by Schrady and Choe.

The model presented here is due to Schrady and Choe. An extension of the model by McCormick (1972) includes constraints on storage volume and probability of depletion. The sensitivity results are easily applied to the extended model. Several assumptions specify the nature of the model. The

first is that all demand which occurs when the on-hand stock is zero, is back-ordered. The model is probabilistic in that the lead time demand is a random variable. Specifically, for the  $i^{th}$  variable, it is assumed that the demand which occurs during the time between the placement of an order and its receipt by the stock point is normally distributed with mean  $\mu_i$  and variance  $\sigma_i^2$ .

For the  $i^{th}$  item, let

- $c_i$  - item unit cost (in dollars),
- $\lambda_i$  - mean demand per unit time (in units),
- $r_i$  - reorder point,
- $Q_i$  - reorder quantity,
- $\Phi(x)$  - the  $\text{Normal}(0,1)$  density function, and
- $\Phi(s)$  - the  $\text{Normal}(0,1)$  complementary cumulative distribution function =  $\int_s^\infty \Phi(x) dx$ .

In addition, let  $K_1$  be the investment limit in dollars, let  $K_2$  be the reorder workload limit and let  $N$  be the total number of items in the inventory. The detailed development of the model is omitted here and the reader is referred to Schrady and Choe (1971) or McCormick (1972). A function needed from that development for the final model is

$$\beta_i(r_i) = \frac{1}{2}(\sigma_i^2 + (r_i - \mu_i)^2)\Phi((r_i - \mu_i)/\sigma_i) - \frac{1}{2}\sigma_i(r_i - \mu_i)\Phi'((r_i - \mu_i)/\sigma_i).$$

Then the general multi-item model of Schrady and Choe is

$$\text{minimize } Z(Q, r) = \sum_{i=1}^N \beta_i(r_i)/Q_i$$

S-C

$$\text{subject to } \begin{aligned} g_1(Q, r) &= k_1 - \sum_{i=1}^N c_i(r_i + Q_i/2 - \mu_i) \geq 0, \\ g_2(Q, r) &= k_2 - \sum_{i=1}^N \lambda_i/Q_i \geq 0, \end{aligned}$$

with  $r_i$  unrestricted,  $Q_i \geq 0$ ,  $i=1, \dots, N$ ,  $Q = (Q_1, \dots, Q_N)^T$  and  $r = (r_1, \dots, r_N)^T$ .

To put the problem in  $(x, \epsilon)$  notation, make the following identifications for  $i=1, \dots, N$ :

$$\begin{aligned} x_{2i-1} &= Q_i, \\ x_{2i} &= r_i, \\ \epsilon_{4i-1} &= \mu_i, \\ \epsilon_{4i} &= \sigma_i, \\ \epsilon_{4i+1} &= c_i, \\ \epsilon_{4i+2} &= \lambda_i. \end{aligned}$$

and  $\epsilon_1 = k_1$ ,  $\epsilon_2 = k_2$ . Rewrite Problem S-C as

$$\text{minimize } f(x, \epsilon) = \sum_{i=1}^N \beta_i(x_{2i}, \epsilon)/x_{2i-1} \quad \text{S-C}(\epsilon)$$

$$\begin{aligned} \text{subject to } g_1(x, \epsilon) &= \epsilon_1 - \sum_{i=1}^N \epsilon_{4i+1}(x_{2i} + x_{2i-1}/2 - \epsilon_{4i-1}) \\ &\geq 0, \\ g_2(x, \epsilon) &= \epsilon_2 - \sum_{i=1}^N \epsilon_{4i+2}/x_{2i-1} \geq 0, \end{aligned}$$

$x_{2i-1} \geq 0$ ,  $x_{2i}$  unrestricted,  $i=1, \dots, N$ , and

$$\begin{aligned} \beta_i(x_{2i}, \epsilon) &= \frac{1}{2}((\epsilon_{4i})^2 + (x_{2i} - \epsilon_{4i-1})^2) \Phi((x_{2i} - \epsilon_{4i-1})/\epsilon_{4i}) \\ &\quad - \epsilon_{4i}(x_{2i} - \epsilon_{4i-1}) \Phi((x_{2i} - \epsilon_{4i-1})/\epsilon_{4i}). \end{aligned}$$

Schrady and Choe consider a three item example. The problem data and the initial starting point for the SUMT program are shown in Table 6.

TABLE 6  
MULTI-ITEM INVENTORY PROBLEM DATA

Item	i=1	i=2	i=3
<b>Data:</b>			
$\mu_i = \epsilon_{4i-1}$	100	200	300
$\sigma_i = \epsilon_{4i}$	100	100	200
$c_i = \epsilon_{4i+1}$	1	10	20
$\lambda_i = \epsilon_{4i+2}$	1,000	1,500	2,000
<b>Starting point:</b>			
$Q_i = x_{2i-1}$	600	270	300
$r_i = x_{2i}$	200	260	400

In addition,  $K_1 = \epsilon_1 = \$8,000$  and  $K_2 = \epsilon_2 = 15$ .

Figure 5 contains the computer output for the final subproblem of a trajectory sensitivity analysis for Problem S-C( $\epsilon$ ) using the data of Table 6. The results indicate that the optimal value function is sensitive to parameters 2, 5, 8, 9, 12, and 13, i.e.,  $K_2$ ,  $c_1$ ,  $\sigma_2$ ,  $c_2$ ,  $\sigma_3$  and  $c_3$  respectively. The fact that the solution is sensitive to the values of the standard deviations of the lead time demand of two items lets the decision maker know that since, these parameters were obtained by sampling, a possible action may be to conduct additional sampling in order to sharpen the estimate of the standard deviation.

The solution value is also very sensitive to all of the item costs. If the structure of Problem S-C is examined, this result is most surprising since the  $c_i$  appear only in the investment constraint and the optimal value function is not very sensitive to the investment limit  $K_1$ . Recall, however,

101100 35 BOUNDARY 0.27499200 -07  
 F= 0.1296000 02 P= 0.1296000 02 M= 0.1296000 02 MAGNITUDE= 0.22017350-02 PHASE= 2  
 THE CURRENT VALUE OF X IS 0.531700 03 0.252780 03 0.2455100 03 0.2770070 03 0.26501760 03 0.43661080 03  
 THE CONSTRAINT VALUES  
 C.1170460-01 0.47966500-04

2ND ORDER ESTIMATES  
 F= 0.1296000 02 P= 0.1296000 02 M= 0.1296000 02 RSIGMA= 0.0 H= 0.0  
 THE CURRENT VALUE OF X IS 0.531700 03 0.2527790 03 0.2455060 03 0.2770060 03 0.26501660 03 0.43661220 03  
 THE CONSTRAINT VALUES  
 -0.13226120-03 NOT INCLUDING THE NON-NEGATIVITIES  
 0.263360-06

1ST ORDER ESTIMATES  
 F= 0.1296000 02 P= 0.1296000 02 M= 0.1296000 02 RSIGMA= 0.0 H= 0.0  
 THE CURRENT VALUE OF X IS 0.531700 03 0.2527790 03 0.2455060 03 0.2770060 03 0.26501660 03 0.43661220 03  
 THE CONSTRAINT VALUES  
 -C.1161460-03 0.32366190-06 NOT INCLUDING THE NON-NEGATIVITIES

LAGRANGE MULTIPLIES  
 F= 0.1296000 02 P= 0.1296000 02 M= 0.1296000 02 RSIGMA= 0.0 H= 0.0  
 THE CURRENT VALUE OF X IS 0.531700-06 0.24105740-06 0.24052230-06 0.22013740-06 0.211413020-06 0.13979300-06  
 THE CONSTRAINT VALUES  
 0.51174570-02 0.62202200-06 NOT INCLUDING THE NON-NEGATIVITIES

SENSITIVITY ANALYSIS  
 THE VALUE OF RSIGMA IS 0.015150-04  
 THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY SILLAGE MADE IS  
 AT X1= 235.6177 X2= 252.7701 AT Y1= 245.5910 AT Z1= 277.0077 X1 Y1 Z1= 205.0374 X1 Z1= 436.6108

PARAMETER	VALUE	DIFFERENCING INTERVAL
1	0.521.03	0.10000-10
2	150.000	0.10000-10
3	100.000	0.10000-10
4	100.210	0.10000-10
5	100.200	0.10000-10
6	100.003	0.10000-10
7	200.000	0.10000-10
8	100.000	0.10000-10
9	100.007	0.10000-10
10	150.217	0.10000-10
11	150.200	0.10000-10
12	200.260	0.10000-10
13	200.000	0.10000-10
14	2700.310	0.10000-10

Fig. 5.—Computer output for Schriady-Choe inventory problem.

ESTATE VALUE FUNCTION SENSITIVITY

FIG. 5<sub>1</sub>—Continued.

that when the partial derivatives are used as sensitivity estimates, one is effectively saying "how much will the objective function (or solution point) change if the parameter is increased by one unit?" Suppose  $c_1$  (parameter 5) is increased by one unit from one to two. Using a linear estimate, the objective function is expected to increase by 2.16 to 15.15 and  $Q_1$  (variable  $x_1$ ) is expected to decrease by 208 units to 325. Figure 6 contains the computer output for the perturbed problem with  $c_1 = 2$ . Note that the value of the optimal value function increased to 14.99 and  $Q_1$  decreased to 411. The changes were quite large and in the directions expected but of course not as large as the linear estimate. Notice, however, that this change in  $c_1$  represented a 100 % increase in the value of the parameter. For purposes of comparison, consider a similar change in the investment constraint  $K_1$  (parameter 1) to which the solution is apparently insensitive. Using the optimal value function sensitivity estimate (-.00517), a 100% increase in the value of the parameter is 8,000 and consequently, a linear estimate of the expected change in the optimal value function would be  $(8,000 \times -.00517) = -41.36$ . The above example does not imply that equivalent percentage changes in parameter values is relevant but rather is meant to emphasize that the sensitivity estimates are valid for a small neighborhood of the given parameter values and as such represent instantaneous changes.

The solution values in Figure 5 are slightly different from those presented in Schraday and Choe (1971). This difference is due to the use of

FIG. 6.—Computer output for perturbed Schrödys-Choe problem.

ESTATE PLANNING

Fig. 6.—Continued.

two different approximations to the complementary cumulative normal distribution. While the solution values vary only slightly with these different estimates of the normal distribution, the variations in the Lagrange multipliers and some sensitivity estimates are greater. For the system library subroutine using the normal distribution error function (erf) to estimate the complementary cumulative distribution, the minimizing trajectory of subproblems showed that the estimate of the Lagrange multipliers deviated from what appeared to be a relatively constant value as the final subproblem was approached. As expected, the optimal value function sensitivity estimates obtained directly from the Lagrangian varied in almost direct proportion with the estimates of the Lagrange multipliers. In Figures 5 and 6, the estimates of the optimal value function sensitivity obtained by the two different methods (chain rule and Lagrangian) are in close agreement. This was not the case using the erf-related subroutine used by Schraday and Choe. This is the same effect experienced in other problems as discussed in Section 4. In this case, however, the  $x$ -derivatives are affected to a slight degree. The major source of this error appears to be the lack of necessary precision associated with the erf-related subroutine for the complementary cumulative normal distribution. The conclusion can only be that one must proceed with caution. From this and other examples, however, it appears that convergence of the Lagrange multiplier estimates along the minimizing trajectory is a good indication that the sensitivity estimates will be accurate.

This large-scale inventory example illustrates a potential "real world" application of sensitivity analysis. It also highlights the need for a careful interpretation of the sensitivity information as well as the recurrent call for caution in the use of a numerical algorithm.

## 6. Conclusions

The purpose of this paper was to present examples of computational implementation of sensitivity analysis with respect to the optimal value function and the Lagrange multipliers using the theoretical results of Fiacco (1973) and Armacost and Fiacco (1975, 1976). All of the problems presented led to new insights into the computational aspects of this type of sensitivity analysis.

The major conclusion is that the sensitivity estimates of the optimal value function obtained directly from the gradient of the Lagrangian, the partial derivatives of the Lagrange multipliers associated with the binding constraints, and to a lesser extent, the partial derivatives of the solution point, are dependent on the accuracy of the estimate of the Lagrange multipliers calculated in the penalty function algorithm and subsequently used in the gradient of the Lagrangian. If the estimates of the Lagrange multipliers along the minimizing trajectory converge to a common value, then it appears that the sensitivity estimates will converge to their true values provided the differencing increment is satisfactory. If, however, the Lagrange multipliers converge and then vary, the sensitivity estimates should be viewed with caution. (Note that the Lagrange multiplier estimates are available in the standard SUMT output and a trajectory analysis is not required.)

Armacost and Fiacco (1974) noted that the differencing increment used in the central differencing formulas was a potential source of error depending on the scaling of the problem. The same caution applies when dealing with Lagrange multiplier and optimal value function sensitivity.

The example problems which included right-hand side parameters indicated that the optimal value function second order sensitivity estimates are themselves very sensitive to the Lagrange multiplier estimates, the differencing increment and the scaling of the problem. It appears that further analysis is needed before this particular program can be used to take advantage of second order sensitivity estimates to improve algorithm performance. This is particularly true for the second partial derivatives of the optimal value function taken with respect to the right-hand side of a binding constraint.

Computer time was provided by The George Washington University Computer Center.

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## APPENDIX A

## SUBROUTINES SENS, LMULT AND PRESEN

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0001          SUBROUTINE SENS
C
C          15 MARCH 1972
C
C THIS VERSION OF THE SENSITIVITY ANALYSIS SUBROUTINE IS USED TO
C COMPUTE THE DIRECTIONAL DERIVATIVES OF X AND F WITH RESPECT TO
C CERTAIN PARAMETERS CODED IN THE ARRAY PAR(1). THE DIRECTIONAL
C DERIVATIVES ARE ESTIMATED FOR ONE PARAMETER AT A TIME WITH NPAR BEING
C THE NUMBER OF PARAMETERS INVOLVED IN THE SENSITIVITY ANALYSIS. THE
C USE OF THE PARAMETERS PAR(2) MUST BE CONSISTENT THROUGHOUT THE
C USER'S SUB-RUTINES.
C THE SUBROUTINE IS USED FOR A SENSITIVITY ANALYSIS AT THE FINAL SUB-
C PROBLEM OR FOR A SENSITIVITY ANALYSIS AT EACH SUBPROBLEM ALONG THE
C MINIMIZING TRAJECTORY. DPAF(20) IS THE ARRAY OF DIFFERENCING
C INTERVALS FOR SPANNING IN THE PARAMETERS PAR(20). DPAH(20) IS
C ASSIGNED VALUES IN SUBROUTINE PARDF.
C THIS APPROACH TO SENSITIVITY ANALYSIS IS DUE TO A. V. FIACCO. THE
C FIRST VERSION WAS CODED BY M. CAUSEY. THE SECOND VERSION WAS CODED
C BY M. C. HALLINGER. THIS IS THE THIRD VERSION WHICH IS AN EXTENSION
C OF THE SECOND VERSION TO MEET SENSITIVITY ANALYSIS ALONG THE
C MINIMIZING TRAJECTORY, AND WAS CODED BY R. L. ABRAHAMS.
0022          IMPLICIT REAL(6.0)
0023          REAL(DA,ENGR),REAL(DPSI,THTAC,EPB1,EPB2
0024          COMMON/DA/DA1,DA2,DA3,DA4,DA5,DA6,DA7,DA8,DA9,DA10,DA11,DA12
0025          COMMON/DPSI/DPSI1,DPSI2,DPSI3,DPSI4,DPSI5,DPSI6,DPSI7,DPSI8,DPSI9
0026          COMMON/INTG/INTG1,INTG2,INTG3,INTG4,INTG5,INTG6,INTG7,INTG8,INTG9
0027          COMMON/VALU/VALU1,VALU2,VALU3,VALU4,VALU5,VALU6,VALU7,VALU8,VALU9
0028          COMMON/TEST/TEST1,TEST2,TEST3,TEST4,TEST5,TEST6,TEST7,TEST8,TEST9
0029          COMMON/SENS/SENS1,SENS2,SENS3,SENS4,SENS5,SENS6,SENS7,SENS8,SENS9,SENS10
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0158          COMMON/DPAR/DPAR1281,DPAR1282,DPAR1283,DPAR1284,DPAR1285,DPAR1286,DPAR1287,DPAR1288,DPAR1289,DPAR1290
0159          COMMON/DPAR/DPAR1291,DPAR1292,DPAR1293,DPAR1294,DPAR1295,DPAR1296,DPAR1297,DPAR1298,DPAR1299,DPAR1300
0160          COMMON/DPAR/DPAR1301,DPAR1302,DPAR1303,DPAR1304,DPAR1305,DPAR1306,DPAR1307,DPAR1308,DPAR1309,DPAR1310
0161          COMMON/DPAR/DPAR1311,DPAR1312,DPAR1313,DPAR1314,DPAR1315,DPAR1316,DPAR1317,DPAR1318,DPAR1319,DPAR1320
0162          COMMON/DPAR/DPAR1321,DPAR1322,DPAR1323,DPAR1324,DPAR1325,DPAR1326,DPAR1327,DPAR1328,DPAR1329,DPAR1330
0163          COMMON/DPAR/DPAR1331,DPAR1332,DPAR1333,DPAR1334,DPAR1335,DPAR1336,DPAR1337,DPAR1338,DPAR1339,DPAR1340
0164          COMMON/DPAR/DPAR1341,DPAR1342,DPAR1343,DPAR1344,DPAR1345,DPAR1346,DPAR1347,DPAR1348,DPAR1349,DPAR1350
0165          COMMON/DPAR/DPAR1351,DPAR1352,DPAR1353,DPAR1354,DPAR1355,DPAR1356,DPAR1357,DPAR1358,DPAR1359,DPAR1360
0166          COMMON/DPAR/DPAR1361,DPAR1362,DPAR1363,DPAR1364,DPAR1365,DPAR1366,DPAR1367,DPAR1368,DPAR1369,DPAR1370
0167          COMMON/DPAR/DPAR1371,DPAR1372,DPAR1373,DPAR1374,DPAR1375,DPAR1376,DPAR1377,DPAR1378,DPAR1379,DPAR1380
0168          COMMON/DPAR/DPAR1381,DPAR1382,DPAR1383,DPAR1384,DPAR1385,DPAR1386,DPAR1387,DPAR1388,DPAR1389,DPAR1390
0169          COMMON/DPAR/DPAR1391,DPAR1392,DPAR1393,DPAR1394,DPAR1395,DPAR1396,DPAR1397,DPAR1398,DPAR1399,DPAR1400
0170          COMMON/DPAR/DPAR1401,DPAR1402,DPAR1403,DPAR1404,DPAR1405,DPAR1406,DPAR1407,DPAR1408,DPAR1409,DPAR1410
0171          COMMON/DPAR/DPAR1411,DPAR1412,DPAR1413,DPAR1414,DPAR1415,DPAR1416,DPAR1417,DPAR1418,DPAR1419,DPAR1420
0172          COMMON/DPAR/DPAR1421,DPAR1422,DPAR1423,DPAR1424,DPAR1425,DPAR1426,DPAR1427,DPAR1428,DPAR1429,DPAR1430
0173          COMMON/DPAR/DPAR1431,DPAR1432,DPAR1433,DPAR1434,DPAR1435,DPAR1436,DPAR1437,DPAR1438,DPAR1439,DPAR1440
0174          COMMON/DPAR/DPAR1441,DPAR1442,DPAR1443,DPAR1444,DPAR1445,DPAR1446,DPAR1447,DPAR1448,DPAR1449,DPAR1450
0175          COMMON/DPAR/DPAR1451,DPAR1452,DPAR1453,DPAR1454,DPAR1455,DPAR1456,DPAR1457,DPAR1458,DPAR1459,DPAR1460
0176          COMMON/DPAR/DPAR1461,DPAR1462,DPAR1463,DPAR1464,DPAR1465,DPAR1466,DPAR1467,DPAR1468,DPAR1469,DPAR1470
0177          COMMON/DPAR/DPAR1471,DPAR1472,DPAR1473,DPAR1474,DPAR1475,DPAR1476,DPAR1477,DPAR1478,DPAR1479,DPAR1480
0178          COMMON/DPAR/DPAR1481,DPAR1482,DPAR1483,DPAR1484,DPAR1485,DPAR1486,DPAR1487,DPAR1488,DPAR1489,DPAR1490
0179          COMMON/DPAR/DPAR1491,DPAR1492,DPAR1493,DPAR1494,DPAR1495,DPAR1496,DPAR1497,DPAR1498,DPAR1499,DPAR1500
0180          COMMON/DPAR/DPAR1501,DPAR1502,DPAR1503,DPAR1504,DPAR1505,DPAR1506,DPAR1507,DPAR1508,DPAR1509,DPAR1510
0181          COMMON/DPAR/DPAR1511,DPAR1512,DPAR1513,DPAR1514,DPAR1515,DPAR1516,DPAR1517,DPAR1518,DPAR1519,DPAR1520
0182          COMMON/DPAR/DPAR1521,DPAR1522,DPAR1523,DPAR1524,DPAR1525,DPAR1526,DPAR1527,DPAR1528,DPAR1529,DPAR1530
0183          COMMON/DPAR/DPAR1531,DPAR1532,DPAR1533,DPAR1534,DPAR1535,DPAR1536,DPAR1537,DPAR1538,DPAR1539,DPAR1540
0184          COMMON/DPAR/DPAR1541,DPAR1542,DPAR1543,DPAR1544,DPAR1545,DPAR1546,DPAR1547,DPAR1548,DPAR1549,DPAR1550
0185          COMMON/DPAR/DPAR1551,DPAR1552,DPAR1553,DPAR1554,DPAR1555,DPAR1556,DPAR1557,DPAR1558,DPAR1559,DPAR1560
0186          COMMON/DPAR/DPAR1561,DPAR1562,DPAR1563,DPAR1564,DPAR1565,DPAR1566,DPAR1567,DPAR1568,DPAR1569,DPAR1570
0187          COMMON/DPAR/DPAR1571,DPAR1572,DPAR1573,DPAR1574,DPAR1575,DPAR1576,DPAR1577,DPAR1578,DPAR1579,DPAR1580
0188          COMMON/DPAR/DPAR1581,DPAR1582,DPAR1583,DPAR1584,DPAR1585,DPAR1586,DPAR1587,DPAR1588,DPAR1589,DPAR1590
0189          COMMON/DPAR/DPAR1591,DPAR1592,DPAR1593,DPAR1594,DPAR1595,DPAR1596,DPAR1597,DPAR1598,DPAR1599,DPAR1600
```

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T-335

LINE NUMBER	FORMAT/COMMENT	PARAMETER VALUE	DIFFERENCING INTERVAL /
0033	50 FORMAT(6SH)	PARAMETER VALUE	017230
0034	1 (14.2E-01,0.5,0.0,0.010,0.01 )		017240
0035	60LT(0.015)		017250
0036	55 FORMAT(//)		017260
C 0037	EVALUATE F, G AND H.		017270
C 0038	CALL RESTITUE,F)		017280
C 0039	IF(EH1M2.GT.0.0) GO TO 85		017290
C 0040	DO A1 J=1,NPHZ		017300
C 0041	CALL RESTITUE,B(J,J))		017310
0042	80 CONTINUE		017320
0043	IF(EH1M2.GT.0.0) GO TO 85		
0044	CALL PRESEN(DU,KTEST)		
C 0045	COMPUTE D,L,F.		017330
0046	05 CALL GRAD(F)		017340
0047	DO 95 I=1,N		017350
0048	K3H1I = DEL(I,I)		017360
0049	95 CONTINUE		017370
C 0050	COMPUTE (DEL(I,I)*2 P - STORED IN A.		017380
0051	CALL SECUND(P2)		017390
C 0052	PFFRQ-N THE L-U DECOMPOSITION OF A.		017400
0053	DO 102 I=1,N		017410
0054	102 CALL ECRG(C)		017420
0055	105 CONTINUE		017430
0056	INPUT1		017440
0057	NT3=1		017450
C 0058	CALL INVERSE(L)		017460
C 0059	CHECK TO MAKE SURE AN ORTHOGONAL MOVE IS NOT ATTEMPTED.		017470
0060	07 11C 1=1,N		017480
0061	IF(DL(K1M1).LT.0.0,C) GO TO 115		017490
0062	115 WRITE(*,110)		017500
C 0063	165 FORMAT(6SH), THE MATRIX OF SECOND PARTIALS IS NOT POSITIVE DEFINITE		017510
0064	166 SENSITIVITY ANALYSIS IS TERMINATED		017520
0065	GO TO 202		017530
0066	11C CONTINUE		017540
0067	DO 110 171,INPAR		017550
C 0068	ANH1I = PAR(I,I)		017560
0069	120 CONTINUE		017570
0070	DO 220 171,INPAR		017580
0071	IF(EH1M2.GT.0.0) GO TO 115		017590
0072	115 IF(EH1M2.GT.0.0) GO TO 205		
0073	205 IF(EH1M2.GT.0.0) GO TO 121		
0074	CALL FMULT(D,DL,DU,DUM1,DUM2)		
C 0075	COMPUTE DIF1L(D1/D2*(J)) AND DIF1/D4*(J) USING CENTRAL DIFFERENCING.		017590
0076	121 PAR(J,I) = PAR(J,I) + DPAR(J,I)		017600
C 0077	CALL RESTINT(D,DI)		017610
0078	122 IF(DI.GT.0.0) GO TO 126		017620
C 0079	07 125 1=1,N		017630
C 0080	CALL C1STAT11,B(J,I))		017640
0081	123 IF(EH1M2.GT.0.0) GO TO 122		017650
C 0082	123 INPAR(J,I) = INDPAR(J,I)		017660
0083	PAR(J,I) = ANH1I(J,I)		017670
0084	07 11F(1,1,124) J,INPAR(J,I)		017680
C 0085	124 FMULT(1,M,RESETTING DPARE,I,12,3M1= .610.0)		017690
0086	IF(DPARE,I,I).LT.0.0) GO TO 201		017700
0087	GO TO 121		017710
0088	125 CONTINUE		017720
0089	126 IF(EH1M2.GT.0.0) GO TO 120		017730
0090	DO 127 1=1,N		017740
C 0091	127 IF(DI.GT.0.0) GO TO 128		017750
C 0092	CALL RESTINT(D,DI)		017760
0093	128 IF(EH1M2.GT.0.0) GO TO 129		017770
C 0094	CALL GRAD(F)		017780
0095	DO 129 1=1,N		017790
C 0096	DL(1,I,I)=DL(1,I,I)		017800
C 0097	130 CONTINUE		017810
0098	07 128 1=1,N		017820
0099	CALL GRAD(F)		017830
C 0100	130 IF(EH1M2.GT.0.0) GO TO 131		017840
0101	CALL RESTINT(D,DI)		017850
C 0102	131 IF(EH1M2.GT.0.0) GO TO 136		017860
0103	DO 132 1=1,N		017870
C 0104	132 CALL C1STAT11,B(J,I))		017880
0105	132 IF(EH1M2.GT.0.0) GO TO 133		017890
0106	133 GO TO 121		017900
C 0107	135 CONTINUE		017910
0108	136 IF(EH1M2.GT.0.0) GO TO 134		017920
0109	DO 137 1=1,N		017930
C 0110	137 IF(DI.GT.0.0) GO TO 138		017940
C 0111	138 CALL RESTINT(D,DI)		017950
0112	138 CONTINUE		017960

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T-335

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0106      130  IF(NEQG<=6,0,1) GO TO 139
0107      CALL LMULT12,DELMU,DEM,DUS
0108  139  CALL G1AC(2)
0109      IF(FIDF>VAL1/DEM
0110      OR 140  THEN
0111      DELTET1=DELTET1 + DELT0111/DEM
0112  140  CONTINUE
0113      PA1(1)=P2HES(1)
0114  C  HAVING PERTURB FACTORED AT SOLVE AREA FOR X0
0115  C  WHERE HODLE AND H0D/H0D(X0).
0116      CALL LMULT12,DEM
0117  C  PRINT OUT LNS/DEM
0118      WRITE(6,150) J
0119  150  FORMAT(1A21) A-DERIVATIVES ARE WITH RESPECT TO PARAMETER ,12)
0120      DD 172  I=1,6,6
0121      I1=MNC(1,9,9)
0122      DELTET1,1721 (1JJJ,DEM)JJJ,DEM,111
0123  160  FORMAT(1A21 DD=12,2H)1,DEM,111
0124  170  CONTINUE
0125      IF(NEQG<=6,0,1) GO TO 175
0126      IF(EV1<0,1) GO TO 321
0127      CALL LMULT12,DELMU,DEM,DUS
0128      DELTET1,1721 J
0129  350  FORMAT(1A21 U-DERIVATIVES WITH RESPECT TO PARAMETER ,12)
0130      DD 171  I=1,6,6
0131      I1=MNC(1,9,9)
0132      DELTET1,1721 (1JJJ,DEM)JJJ,DEM,111
0133  352  FORMAT(1A21 DD=12,2H)1,DEM,111
0134  351  CONTINUE
0135  351  IF(NEQG<=6,1) GO TO 375
0136      CALL LMULT12,DELMU,DEM,DUS
0137      DELTET1,1721 J
0138  360  FORMAT(1A21 U-DERIVATIVES WITH RESPECT TO PARAMETER ,12)
0139      DD 361  I=1,6,6
0140      I1=MNC(1,9,9)
0141
0142      DELTET1,1721 (1JJJ,DEM)JJJ,DEM,111
0143  362  FORMAT(1A21 DD=12,2H)1,DEM,111
0144  361  CONTINUE
0145  375  CONTINUE
0146  C  COMPUTE OF/DDEL12.
0147      DD 143  I=1,4
0148      DD 144  I=1,4*H12*DFLC(1)
0149  182  CONTINUE
0150  C  PRINT DFLC(1).
0151      A = FLTC(1,1901) DF
0152  187  FORMAT(1A21,13HDF(FLTC(1))DF,1,G10.6/10M 0000000000/1)
0153  200  CONTINUE
0154      GO TO 202
0155  201  DELTET1,1721 J
0156  106  PRINT(1A21,13HINTILINATING PARAMETER,13,10M DUE TO DFLC = 0 ) /1
0157      GO TO 202
0158  202  NT=44H
0159      CALL DFLC
0160      DD 205  I=1,4
0161      DELTET1 = DELTET1
0162  295  DELTET1 = DELTET1
0163      RETURN
0164
0165

```

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```

0001      SUBROUTINE LMULTIEND(DELBUD,DEM,DUF)
0002      IMPLICIT REAL*8(A-H,O-Z)
0003      REAL*8 PHIN,RATIO,PHSE,THETAC,XFPI,XFPI2
0004      COMMON/SHARF/X2(2),X3(1,2,1),X3(2,2),X3(3,3),X3(4,4),X3(5,5),X3(6,6)
0005      COMMON/VALUF/F,G,P,LSIGMA,LJ1C01,PHD
0006      COMMON/FOALX/M,M1,M2
0007      COMMON/CNST/DFLA(20),DFLAY(20),PHOIN,RATIO,EPST1,THETA0,
0008      INTCTH,NUVINE,X1(1,1),X2(2,2),X3(2,3),X3(3,3),XPI(20),PRL,
0009      ZPL(2,4),PL(4,4),DCTT,PLRAD(20),DTAG(20),
0010      SPFLV,ADFLX,NSIGL,G1,NPHASE,NSATES
0011      DIMENSION DFLA(40),NUV(60)
0012      MHZ = N + Hz
0013      GO TO E1,E2,E3,E4, IND
C IND = 1
0014      DO 100 I=1,MHZ
0015      100  DELNUC(I) = 0.
0016      RETURN
C IND = 2
0017      2     DO 50 J=1,MHZ
0018      50    DELNUC(J) = (DELNUC(J) - DFLA(J))/DEM
0019      RETURN
0020      3     DO 70 K=1,MHZ
0021      70    CALL GHAD(K)
0022      CALL FESTNTE(K,VAL)
0023      SUM = 0.
0024      70    SUM = SUM + DELNUC(K)*VAL
0025      IF(IND.EQ.0.01) GC TO 10
0026      DFLA(1) = SUM + D(LMULTIEND,0.002)
0027      GC TO 70
0028      60    DFLA(1) = 2.0*(SUM + DELNUC(1))/DEM
0029      70    CONTINUE
0030      RETURN
0031      END
0032

```

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```

0031      SUBROUTINE PREGENRHO(KTEST)
0032      IMPLICIT REAL*8(A-H,L-Z)
0033      COMMON/CHARL/X(20),DEL(20),A(20,20),N,M,MN,NPL,NM1
0034      COMMN/IN/VALUE/F,G,F0,ESIGMA,FJ(40),RHO
0035      COMMN/OUT/NPAR(20),DPAF(20),NPAR,ISENS
0036      DIMENSION GX(AC),OU(40),KTEST(20),KLST(20)
0037      MPHZ = M + MZ
0038      FTST = G(0,0) + DAHS(F)
0039      DO 120 J=1,NPAR
0040      KTEST(J) = 0
0041      PAR(J) = PAR(J) + DPAR(J)
0042      CALL FESTNT(0,DF)
0043      IF(MHZ>F,0) GO TO 20
0044      DO 17  I=1,MPHZ
0045      CALL FESTNT(I,OU(I))
0046      CONTINUE
0047      10  DFY = 2. * DPAF(J)
0048      HAL(J) = PAR(J) - DEM
0049      CALL FESTNT(0,DF)
0050      IF(MHZ>F,0) GO TO 40
0051      DO 32  I=1,MPHZ
0052      CALL FESTNT(I,GX(I))
0053      CONTINUE
0054      30  DFPS = (DF - XF)/DEM
0055      IF(MHZ>F,0) GO TO 60
0056      DO 50  I=1,MPHZ
0057      OU(I) = (OU(I) - GX(I))/DEM
0058      SUM = DFPS
0059      60  IF(I>0,0) GO TO 80
0060      DC = 7; I=1,N
0061      70  SUM = SUM - RMC/FJ(I)*OU(I)
0062      IF(I>F,0) GO TO 95
0063      TSUM = 0.
0064      DO 90  I=1,MPHZ
0065      IM = I+M
0066      90  TSUM = TSUM + FJ(IM)*OU(IM)
0067      SUM = SUM + TSUM + 2./RMC
0068      DEL(J) = SUM
0069      PAR(J) = PAR(J) + DPAR(J)
0070      DTEST = DAHS(DFL(J))
0071      IF(DTEST.GE.FTEST) KTEST(J) = 1
0072      CONTINUE
0073      100  WRITE(6,F01) ((J,DEL(J)), JJ=1,11)
0074      600  FORMAT(12X,14HOPTIMAL VALUE FUNCTION SENSITIVITY    //)
0075      DO 250  J=1,NPAR,5
0076      I=FJ(0,J)
0077      WRITE(6,F02) ((J,DEL(J)), JJ=1,11)
0078      601  FORMAT(14H DF/DA(.12,2H)=,G14.7)
0079      200  CONTINUE
0080      JJ = 0
0081      DO 250  J=1,NPAR
0082      IF(KTEST(J).EQ.0) GO TO 250
0083      JJ = JJ + 1
0084      KLST(JJ) = J
0085      CONTINUE
0086      IF(JJ.EQ.0) GO TO 300
0087      WRITE(6,F02)
0088      602  FORMAT(15H DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS  )
0089      WRITE(6,F02) ((J,DEL(J)), I=1,JJ)
0090      603  FORMAT(15H A5(12,2H), JJ)
0091      WRITE(6,F04)
0092      604  FORMAT(/)
0093      IF(JJ.NE.0) GO TO 300
0094      WRITE(6,F05)
0095      300  WRITE(6,F05)
0096      605  FORMAT(15H THERE ARE NO DETAILED SENSITIVITY RESULTS    //)
0097      RETURN
0098      END

```

## APPENDIX B

## USER SUBROUTINES FOR SCHRADY-CHOE PROBLEM

```

0031      SUBROUTINE READIN
0032      IMPLICIT REAL*8(A-H,O-Z)
0033      COMMON/INV/BETA(20),PHI(20),DENSE(20),IDENT(20),NI
0034      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
0035      901  FORMAT(1S,4F12.6)
0036      READ(5,901) NI,PAR(1),PAR(2)
0037      WRITE(6,901) NI,PAR(1),PAR(2)
0038      DO 100 I=1,NI
0039      READ(5,901) IDENT(I),(PAR(4*I-2+J),J=1,4)
0040      WRITE(6,901) IDENT(I),(PAR(4*I-2+J),J=1,4)
0041      100  CONTINUE
0042      NPAR = 4*NI+2
0043      RETURN
0044      END

0048      SUBROUTINE RESTN(IN,VAL)
0049      IMPLICIT REAL*8(A-H,O-Z)
0050
0051      COMMON/SHAPE/X(20),DFL(20),A(20,20),N,M,MN,NP1,NM
0052      COMMON/N/INV/BETA(20),PHI(20),DENSE(20),IDENT(20),NI
0053      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
0054      VAL = 0.
0055      IF(IN,F0.0) GO TO 300
0056      IF(IN,F0.1) GO TO 100
0057      200  DO 250 I=1,NI
0058          IJ = 2*I-1
0059          QO = X(IJ)
0060          IF(QO,LT,0.) GO TO 180
0061          VAL = VAL + PAR(4*I+2)/QO
0062          VAL = PAR(2) - VAL
0063          RETURN
0064          180  VAL = -1.0
0065          RETURN
0066          100  DO 150 I=1,NI
0067              IJI = 2*I
0068              IJ = IJI - 1
0069              RR = X(IJ)
0070              IF(RR,LT,0.) GO TO 180
0071              QO = X(IJI)
0072              IF(QO,LT,0.) GO TO 180
0073              VAL = VAL + PAR(4*I+1)*(RR+QO/2.-PAR(4*I-1))
0074              VAL = PAR(1) - VAL
0075              RETURN
0076          300  DO 350 I=1,NI
0077              IJI = 2*I
0078              IJ = IJI - 1
0079              RR = X(IJ)
0080              UU = X(IJI)
0081              UU = PAR(4*I-1)
0082              SS = PAR(4*I)
0083
0084              DELTA = RR - UU
0085              ZH = DELTA / SS
0086              CALL ANUTR(N,FL,UDEN)
0087              PHI(I) = PI
0088              DLNPI(I) = PI
0089              BFTAC(I) = 9.99*(SS+SS*DELTA*DELTA)*PHI(I)-SS*DELTA*DENSE(I)
0090              VAL = VAL + BFTAC(I)/QO
0091              RETURN
0092              END

```

```

0001      SUBROUTINE GRADL(IN)
0002      IMPLICIT REAL*8(A-H,O-Z)
0003      COMMON/SHARE/X(20),DFL(20),A(20,20),N,M,MM,NPI,NMI
0004      COMMON/INV/DETA(20),PHI(20),DENSE(20),IDENT(20),NI
0005      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
0006      IF(INF0.0) GO TO 300
0007      IF(IN,F0.1) GO TO 100
0008      DO 250 I=1,NI
0009      IJ = 2*I
0010      IJ = IJ - 1
0011      OO = X(IJ)
0012      DEL(IJ) = 0.
0013      250 DFL(IJ) = PAR(4+I+2)/OO/OO
0014      RETURN
0015      100 DO 150 I=1,NI
0016      IJ = 2*I
0017      IJ = IJ - 1
0018      DFL(IJ) = -PAR(4+I+1)
0019      150 DFL(IJ) = DFL(IJ)/2.
0020      RETURN
0021      200 DO 350 I=1,NI
0022      IJ = 2*I
0023      IJ = IJ - 1
0024      OO = X(IJ)
0025      RR = X(IJ)
0026
0027      UU = PAR(4+I-1)
0028
0029      SS = PAR(4+I)
0030
0031      DELTA = RR - UU
0032      ZN = DELTA / SS
0033      CALL ANDR(ZN,FI,DEN)
0034      PHI(I) = FI
0035      DENS(I) = DEN
0036      DETA(I) = 0.5*(SS*SS+DELTA*DELTA)*PHI(I)-SS*DELTA*DENSE(I)
0037      DEL(IJ) = (DELTA+PHI(I)-SS*DENSE(I))/OO
0038      350 DEL(IJ) = -DETA(I)/OO/OO
0039      RETURN
0040      END

0041      SUBROUTINE MATRIE(IN,IKK)
0042      IMPLICIT REAL*8(A-H,O-Z)
0043      COMMON/SHARE/X(20),DFL(20),A(20,20),N,M,MM,NPI,NMI
0044      COMMON/INV/DETA(20),PHI(20),DENSE(20),IDENT(20),NI
0045      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
0046      IF(INL0.0) GO TO 300
0047      IF(IN,F0.1) GO TO 100
0048      DO 250 I=1,NI
0049      IJ = 2*I-1
0050      OO = X(IJ)
0051      250 A(IJ,IJ) = -2.*PAR(4+I+2)/OO*OO
0052      RETURN
0053      100 IKK = 1
0054      RETURN
0055      300 DO 750 I=1,NI
0056      IJ = 2*I
0057      IJ = IJ - 1
0058      OO = X(IJ)
0059      RR = X(IJ)
0060
0061      UU = PAR(4+I-1)
0062
0063      SS = PAR(4+I)
0064
0065      DELTA = RR - UU
0066      ZN = DELTA / SS
0067      CALL ANDR(ZN,FI,DEN)
0068      PHI(I) = FI
0069      DENS(I) = DEN
0070      DETA(I) = 0.5*(SS*SS+DELTA*DELTA)*PHI(I)-SS*DELTA*DENSE(I)
0071      A(IJ,IJ) = 2.*DETA(I)/OO*OO
0072      A(IJ,IJ) = -DETA(I)-PHI(I)-SS*DENSE(I)/OO/OO
0073      A(IJ,IJ) = PHI(I)/OO
0074      750 RETURN
0075      END

0076      SUBROUTINE ANDR(XK,PHI,DENSE)
0077      IMPLICIT REAL*8(A-H,C-Z)
0078      AX = 0.435(XK)
0079      T = 1.0/(1.+XK+2.316419*XK)
0080      DENS = (C-0.007622201*XP(-XK+XK/2.))
0081      PHI = DENS*STO((1.37027607-1.021256)*T+1.781470)*T
0082      S = C-0.366670107 + 0.31970151
0083      IF(XK) 1,2,2
0084      1    PHI = 1.0 - PHI
0085      2    RETURN
0086      END

```

## APPENDIX C

## SHELL PROBLEM DATA

$i \backslash j$	1	2	3	4	5	
$c_{ij}$	-15	-27	-36	-18	-12	
$c_{ij}$	1	30	-20	-10	32	-10
$c_{ij}$	2	-20	39	-6	-31	32
$c_{ij}$	3	-10	-6	10	-6	-10
$c_{ij}$	4	32	-31	-6	39	-20
$c_{ij}$	5	-10	32	-30	-20	30
$d_j$		4	8	10	6	2
$a_{ij}$	1	-16	2	0	1	0
$a_{ij}$	2	0	-2	0	0.4	2
$a_{ij}$	3	-3.5	0	.	0	0
$a_{ij}$	4	0	-2	0	-4	-1
$a_{ij}$	5	0	-9	-2	1	-2.8
$a_{ij}$	6	2	0	-4	0	0
$a_{ij}$	7	-3	-1	-1	-1	-1
$a_{ij}$	8	-1	-2	-3	-2	-1
$a_{ij}$	9	1	2	3	4	5
$a_{ij}$	10	1	1	1	1	1
						$b_1$

## CATTLE FEED PROBLEM DATA

j	$c_j$	$a_j$	$\mu_j$	$\sigma_j$	
1 Barley	24.55	2.3	12.0	0.53	(parameter 1)
2 Oats	26.75	5.6	11.9	0.44	(parameter 2)
3 Sesame Flakes	39.00	11.1	41.8	4.50	(parameter 3)
4 Groundnut Meal	40.00	1.3	52.1	0.79	(parameter 4)
$\phi = -1.645$ (parameter 5) corresponds to probability of 0.95					
$p_m = 21$ (parameter 6)					
$o_{ln} = 5$ (parameter 7)					

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